ONE PERSON – ONE VOTE is an democratic idea of equality

But what if the voters are not PEOPLE but are governments? countries? states?
If the institutions are not equal, then the number of votes they control should not be equal.

This situation where each voter is not equal in the number of votes they control is called:

2.1 Weighted Voting

Important terms:

_________________________________________: A voting situation where voters are not necessarily equal in the number of votes they control.

______________: A vote with only two choices. (usually yes/no)

______________: The voters (symbolized by P₁, P₂, P₃, etc.)

______________: The number of votes a player controls.

______________: The smallest number of votes required to “pass” a motion.
Notation

\[ q: w_1, w_2, w_3, \ldots, w_n \] \quad q = \quad w's =

Example 1.  \[14: 8, 6, 5, 1\]

quota = \quad Player 1 (P_1) = controls ___ votes /“has a weight of ____”
total votes = \quad Player 2 (P_2) = controls ___ votes
\quad Player 3 (P_3) = controls ___ votes
\quad Player 4 (P_4) = controls ___ vote

Example 2.  Given the weighted voting system \[16: 8, 6, 4, 4, 3, 1\], state the following:

The number of players (N): ____ \quad The total number of votes (V): ____
The weight of P_5: ____ \quad The minimum % of the quota to nearest whole %: ____

Common Types of Quotas:

Simple majority/Strict majority \[\rightarrow\]
Two-thirds majority \[\rightarrow\]
Unanimity \[\rightarrow\]

Example 3. A group of friends decide to start a small business. For every $100 they invest in the business, they receive one vote. The weights of the friends are shown in the table. They decide that the quota will be set at a simple majority or more of the votes.

\begin{tabular}{|l|c|}
\hline
Name & Votes \\
\hline
Susan & 14 \\
Peter & 10 \\
Edmund & 25 \\
Lucy & 18 \\
Tumnus & 5 \\
\hline
\end{tabular}

a) What is the total number of players, N?

b) What is the total number of votes, V?

c) Describe as a weighted voting system using standard notation.
Weighted Voting Issues

Example 4:
Four partners decide to start a business. \( P_1 \) buys 8 shares, \( P_2 \) buys 7 shares, \( P_3 \) buys 3 shares and \( P_4 \) buys 2 shares. One share = one vote.

\[ a. \] The quota is set at 13 votes. Describe as a weighted voting system in standard notation.

\[ b. \] The partnership above decides the quota is too high and changes the quota to 10 votes.

\[ c. \] The partnership above decides to make the quota equal to 21 votes.

For a weighted voting system to be legal: the quota must be at least a _______________ and no more than _______________.

Symbolically: If \( V = w_1 + w_2 + w_3 + \ldots + w_n \), then \( \frac{V}{2} < q \leq V \).

\[ d. \] What if our partnership changed the quota to 19?

Example 5. \[ q: 7, 2, 1, 1, 1 \]

\[ a) \] What is the smallest value that the quota can take? _____

\[ b) \] What is the largest value that the quota can take? _____
Example 6: \[q: 36, 32, 8, 8, 4\]

c) What is the value of the quota if \textit{at least three-fourths} of the votes are required to pass a motion?

\textit{more than three-fourths} of the votes are required?

**Dictators, Dummies, and Veto Power**

VOCAB: A ____________ is a player whose weight is greater than or equal to the quota.

Example 7: \[11: 12, 5, 4\]

\[P_1\text{ has all the power} \Rightarrow \]

\[P_2\text{ and } P_3\text{ have no power} \Rightarrow \]

VOCAB: A player has ____________________ if they are not a dictator, but they can keep the other players from combining their votes to meet the quota. Not every weighted voting system has someone with veto power. Sometimes MANY players can have veto power.

Example 8: \[30: 10, 10, 10, 9\]

Example 9: \[12: 9, 5, 4, 2\]

Note:
If any player is a dictator, then EVERY OTHER PLAYER is a dummy.

Even if there is no dictator, there may still be dummies.
Example 11. Determine which players, if any, are dictators or have veto power.

a) [15: 16, 8, 4, 1]  
b) [18: 16, 8, 4, 1]  
c) [24: 16, 8, 4, 1]

Example 12. Consider [q: 8, 7, 2]. Find the smallest value of q for which

a) all three players have veto power  
b) $P_2$ has veto power, but $P_3$ does not

2.2/2.4A The Banzhaf Power Index/Counting SUBSETS

Who is the most powerful player?

________________________: A group of players who choose to vote together {A SUBSET OF THE VOTERS}

_______________________________: The set of all voters. This represents a unanimous vote.

Weight of the coalition:

Winning coalitions—

Losing coalitions—

________________________: Any player who MUST BE PRESENT in a winning coalition in order for it to remain a winning coalition.

Note:
If you subtract the critical player's votes from the coalition, the number of votes drops below the quota.
Example 1: Find the critical player or critical players in each of the following coalitions.

[15: 13, 9, 5, 2]

a) \( \{P_1, P_4\} \)  

b) \( \{P_2, P_3, P_4\} \)  

c) \( \{P_3, P_4\} \)  

d) \( \{P_1, P_2, P_3\} \)

[51: 30, 25, 20]

a) \( \{P_1, P_3\} \)  

b) \( \{P_1, P_2, P_3\} \)  

c) \( \{P_2, P_3, P_4\} \)  

d) \( \{P_2, P_3\} \)  

The Banzhaf Power Index: A player’s power is proportional to the number of coalitions for which that player is critical. The more often a player is critical, the more power he holds.

Example 2: A weighted voting system with three players has the following winning coalitions (with critical players underlined). Find the Banzhaf Power Distribution for the weighted voting system.

\( \{R_1, R_2\}, \{P_1, P_3\}, \text{ and } \{R_1, R_2, P_3\} \).

**TO FIND THE BANZHAF POWER DISTRIBUTION:**

1. Write down all winning coalitions.

2. Underline (circle/highlight) the critical player(s) in each winning coalition.

3. Determine the total critical count, \( T \) (total number of underlines)

4. Determine each player’s critical count, \( B_n \)

5. Banzhaf Power = \( \frac{B_n}{T} \)
Counting Subsets:

How do you know you have all the possible coalitions written down?

<table>
<thead>
<tr>
<th>Table 2.3 The Subsets of a Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set</td>
</tr>
<tr>
<td>Number of subsets</td>
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<tr>
<td>Subsets</td>
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</tbody>
</table>

If \( n = \) number of players in a weighted voting system, then the number of possible subsets is: \( 2^n - 1 \)

Example 3: Find the Banzhaf Power index for the weighted voting system: \([101: 99, 98, 3]\)

Banzhaf Coalitions: 3 Players

\[
\begin{align*}
\{P₁\} & \quad \{P₁, P₂\} & \quad \{P₁, P₂, P₃\} \\
\{P₁\} & \quad \{P₁, P₃\} \\
\{P₁\} & \quad \{P₂, P₃\} \\
\{P₂\} & \quad \{P₁, P₃\} \\
\{P₃\} & \quad \{P₁, P₃\}
\end{align*}
\]
Example 4: Find the Banzhaf Power Distribution for [40: 30, 20, 10]

Example 4: Find the Banzhaf Power Distribution for [6: 4, 3, 2, 1]

Example 5: Consider the weighted voting system. Find the Banzhaf Power Distribution of this weighted voting system when:

a) [15: 10, 8, 6, 4]

b) [12: 12, 8, 1, 1]

What I expect to see for “work” on your homework:

1. Write down all possible coalitions and cross off losers OR just the winning coalitions.
2. Critical Players should be circled or underlined.
3. Show fraction of BPI for each player AND calculate the % for BPD.
**Where weighted voting systems/Banzhaf are used:**

Banzhaf is used to QUANTIFY the amount of power each player holds.

1. **Nassau County Board of Supervisors** (see p. 55): Votes were given to districts according to population and quota was simple majority. \([58: 31, 31, 28, 21, 2, 2]\) Banzhaf showed that two of the six counties actually had no voting power—that they were actually dummy voters. Final result: 1993 court decision abolishing weighted voting in New York States. “Districts” were created of roughly the same population and each given one voted.

2. **United Nations Security Council**: Banzhaf shows that a permanent member of the council holds more than 10 times the amount of power as one of the non-permanent members. There are 5 permanent members (Britain, China, France, Russia, US) and 10 non-permanent members. This voting arrangement may change as others are being considered for permanent membership.

3. **European Union** Banzhaf quantifies the amount of power each nation has and shows that smaller nations such as Luxembourg and Malta still hold some power.

<table>
<thead>
<tr>
<th>TABLE 2-8</th>
<th>The European Union (2008): Banzhaf Power Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Member nation</td>
<td>Weight</td>
</tr>
<tr>
<td>France, Germany, Italy, United Kingdom</td>
<td>29</td>
</tr>
<tr>
<td>Spain, Poland</td>
<td>27</td>
</tr>
<tr>
<td>Romania</td>
<td>14</td>
</tr>
<tr>
<td>Netherlands</td>
<td>13</td>
</tr>
<tr>
<td>Belgium, Greece, Czech Republic, Hungary, Portugal</td>
<td>12</td>
</tr>
<tr>
<td>Austria, Bulgaria, Sweden</td>
<td>10</td>
</tr>
<tr>
<td>Denmark, Ireland, Lithuania, Slovakia, Finland</td>
<td>7</td>
</tr>
<tr>
<td>Cyprus, Estonia, Latvia, Luxembourg, Slovenia</td>
<td>4</td>
</tr>
<tr>
<td>Malta</td>
<td>3</td>
</tr>
</tbody>
</table>
2.4/2.5 The Shapley Shubik Power Index:

**The Shapley-Shubik Power Index:**

A player’s power is proportional to the number of **sequential-coalitions** for which that player is **pivotal**. The more times a player is pivotal, the more power he holds.

**Sequential coalition:**

<table>
<thead>
<tr>
<th>Sequential Coalitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle P_1, P_2, P_3 \rangle )</td>
</tr>
<tr>
<td>( \langle P_1, P_3, P_2 \rangle )</td>
</tr>
<tr>
<td>( \langle P_2, P_1, P_3 \rangle )</td>
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<tr>
<td>( \langle P_3, P_2, P_1 \rangle )</td>
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<tr>
<td>( \langle P_2, P_3, P_1 \rangle )</td>
</tr>
<tr>
<td>( \langle P_3, P_1, P_2 \rangle )</td>
</tr>
</tbody>
</table>

**Banzhaf:** \( \{ P_1, P_2, P_3 \} \)

These 3 players decide to vote together. They form a coalition. Order listed in the \( \{ \} \) doesn’t matter.

**Shapley-Shubik:** \( \langle P_1, P_3, P_2 \rangle \)

These 3 players decide to vote together. \( P_1 \) votes 1\textsuperscript{st}, \( P_3 \) votes 2\textsuperscript{nd}, \( P_2 \) votes 3\textsuperscript{rd}. They form a sequential coalition. Order listed in the \( \langle \rangle \) is important.

**Pivotal player:**

**Example 1: Find the Pivotal Player**

Given the weighted voting system \([5: 3,2,1,1]\) find the pivotal player for the given sequential coalition.

a) \([P_1, P_4, P_3, P_2]\)  
b) \([P_3, P_1, P_2, P_4]\)  
c) \([P_4, P_3, P_2, P_1]\)


Counting Sequential Coalitions:
A sequential coalition is really just a permutation of
List the possible sequence for 3 players. How many are there?

How many sequential coalitions are there for 5 players?

Multiplication Rule:
If there are \( m \) ways to do task 1 and \( n \) ways to do task 2, then there are \( mn \) ways to do both tasks together.

Factorials:
If \( N \) = the number of players, then the number of sequential coalitions is \( N! \)
\[
N! = N \times (N-1) \times \ldots \times 3 \times 2 \times 1
\]

Shapley-Shubik Power Distribution

Example 2. A weighted voting system has three players. The sequential coalitions are listed below with the pivotal player underlined in each. Find the Shapley-Shubik Power Distribution for the weighted voting system.

\[
< P_1, P_2, P_3 >, < P_1, P_3, P_2 >, < P_2, P_1, P_3 >, \\
< P_1, P_3, P_2 >, < P_2, P_1, P_3 >, < P_3, P_1, P_2 >
\]

Example 3. A weighted voting system has four players. The sequential coalitions are listed below with the pivotal player underlined in each. Find the Shapley-Shubik Power Distribution for the weighted voting system.

\[
< P_1, P_2, P_3, P_4 >, < P_2, P_1, P_3, P_4 >, < P_3, P_1, P_2, P_4 >, < P_4, P_1, P_2, P_3 >, \\
< P_1, P_2, P_3, P_4 >, < P_2, P_1, P_3, P_4 >, < P_3, P_1, P_2, P_4 >, < P_4, P_1, P_2, P_3 >, \\
< P_1, P_2, P_3, P_4 >, < P_2, P_1, P_3, P_4 >, < P_3, P_1, P_2, P_4 >, < P_4, P_1, P_2, P_3 >, \\
< P_1, P_4, P_2, P_3 >, < P_2, P_4, P_1, P_3 >, < P_3, P_4, P_1, P_2 >, < P_4, P_3, P_2, P_1 >, \\
< P_1, P_4, P_3, P_2 >, < P_2, P_4, P_3, P_1 >, < P_3, P_4, P_2, P_1 >, < P_4, P_3, P_2, P_1 >
\]

TO FIND THE Shapley-Shubik POWER DISTRIBUTION:

1. Write down all \( N! \) sequential coalitions.
2. Underline (circle/highlight) the pivotal player(s) in each winning coalition.
3. Determine each player’s pivotal count, \( SS_n \)
4. Shapley Shubik Power = \( \frac{SS_n}{N!} \)
Example 4: Find the Shapley Shubik Power Distribution for [4: 3, 2, 1]

Example 5: Find the Shapley-Shubik Power Distribution for [6: 4, 3, 2, 1]

Example 6: Find the Shapley-Shubik Power Distribution for [15:10, 8, 4, 2]