Chapter 3: Probability

3.1: Basic Concepts of Probability and Counting

**Probability** measures how ________________ an event is to occur.
Probability is measured between 0 and 1 and are frequently measured as fractions, decimals, and percents.

\[ P(A) \] is read “P of A,” and denotes the probability of event A.

If:
\[ P(A) = 1 \]
\[ P(A) = 0 \]

A is _______________ to occur. A _______________ to occur.

**Probability Assignments – Three Types:**

1. **INTUITION/PERSOAL/SUBJECTIVE** Probability assignment based on intuition incorporates past ________________, ________________, or ________________ to estimate the likelihood of an event.

2. **RELATIVE FREQUENCY/EXPERIMENTAL/EMPIRICAL**

Probability based on the formula:

\[ P(\text{Event}) = \text{relative frequency} = \frac{f}{n} \]

where \( f \) is the frequency of the event in a sample of \( n \) observations.

3. **EQUALLY LIKELY OUTCOMES/CLASSICAL/THEORETICAL**

Probability based on the formula:

\[ P(\text{event}) = \frac{\text{Number of outcomes favorable to event}}{\text{Total number of outcomes}} \]
Example 1: Consider each of the following events, and determine how the probability is assigned.

**Subjective** (Personal opinion)? **Empirical** (Relative Frequency)? **Classical** (Equally Likely Outcomes)?

a) Henry figures that if he guesses on a true-false question, the probability of getting it right is 0.50.

b) A sports announcer claims that Sheila has a 90% chance of breaking the record in the 100-yard dash.

c) The Right to Health Lobby claims that the probability of getting an erroneous medical laboratory report is 0.40, based on a random sample of 200 laboratory reports, of which 80 were erroneous.

d) A random sample of 500 students at Hudson College were surveyed, and it was determined that 375 wore glasses or contact lenses. Estimate the probability that a Hudson College student selected at random wears corrective lenses.

e) The Friends of the Library host a fund raising barbecue. George is on the cleanup committee. There are four members on this committee, and they draw lots to see who will clean the grills. What is the probability that George will be assigned the cleaning job?

f) Joanna photographs whales for Sea Life Adventure Films. On her next expedition, she is to film blue whales feeding. Based on her knowledge of the habits of blue whales, she is almost certain she will be successful. What specific number do you suppose she estimates for the probability of success?

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**Empirical Probability** vs. **Classical Probability**

Based on ACTUAL data, things you count... Based on THEORETICAL data, things you think about...

Also Called: Experimental Probability Also Called: Theoretical Probability

Formula: Relative Frequency Formula: Equally Likely Outcomes

Example:
You flip a coin 50 times and get heads 28 times. You flip a coin.

P(head) =  
P(head) =
The Law of Large Numbers vs. The Law of Averages

**Law of Large Numbers**: In the long run, as the sample size increases and increases, the relative frequencies of outcomes get closer and closer to the theoretical (or actual) probability value.

*Example:* As I flip a coin repeatedly, as the number of flips becomes very large, the relative frequency of heads will get closer and closer to 50%, the classical probability of getting heads.

**Law of Averages**: An idea that future events are likely to balance out any deviation from a past average.

*Example:* I just flipped 15 heads in a row! I think the next time I flip a coin, I will get tails!

THE LAW OF LARGE NUMBERS IS GOOD MATH. THE LAW OF AVERAGES IS BAD MATH.

Probability doesn’t “OWE” you anything! Try to stop thinking of things in terms of the non-mathematical law of averages and instead focus on the law of large numbers.

*Example 3:* Toss a coin repeatedly. The relative frequency gets closer and closer to $P(\text{head}) = 0.50$.

<table>
<thead>
<tr>
<th>Relative Frequency</th>
<th>0.52</th>
<th>0.518</th>
<th>0.495</th>
<th>0.503</th>
<th>0.4996</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f = \text{number of heads}$</td>
<td>104</td>
<td>259</td>
<td>495</td>
<td>1006</td>
<td>2498</td>
</tr>
<tr>
<td>$n = \text{number of flips}$</td>
<td>200</td>
<td>500</td>
<td>1000</td>
<td>2000</td>
<td>5000</td>
</tr>
</tbody>
</table>

**Important Terminology:**

- **Statistical/Random Experiment**: any random activity that results in a definite outcome
- **Outcome**: The result of a single trial of an experiment.
- **Sample Space**: the set of all outcomes
- **Event**: A subset of the sample space

**Counting Sample Space**

How many items are in the sample space for the probability experiment listed below?

Give an example of a member of the sample set.

*Example 4:*

- a) Guessing the month of a person’s birthday.
- b) Rolling a standard die.
- c) Rolling a pair of dice. (or one die, two times)
- d) Guessing someone’s four-digit PIN number. (assume numbers can repeat)
- e) Drawing a card from a standard deck of 52 playing cards and then without replacing it, drawing a second card.
Example 5: Assign a probability to each following event based on the information provided.

a) A random sample of 400 students at a college were surveyed and 300 of them liked the food in the cafeteria. Estimate the probability that a student randomly selected at that college likes the cafeteria food.

b) Toss a fair die. What is the probability that you get a number less than or equal to 5?

c) Randomly toss two fair coins. What is the sample space of this experiment? What is the probability that you get two heads? What is the probability of at least one head?

Example 6: Isabel Briggs Myers was a pioneer in the study of personality types. The personality types are broadly defined according to four main preferences. Do married couples choose similar or different personality types in their mates? A sample was collected and the results are shown in the table at the right.

<table>
<thead>
<tr>
<th>Number of Similar Preferences</th>
<th>Number of Married Couples</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Four</td>
<td>34</td>
</tr>
<tr>
<td>Three</td>
<td>131</td>
</tr>
<tr>
<td>Two</td>
<td>124</td>
</tr>
<tr>
<td>One</td>
<td>71</td>
</tr>
<tr>
<td>None</td>
<td>15</td>
</tr>
</tbody>
</table>

a) Use the data to estimate the probability that they will have All Four preferences in common.

b) Find the probability of having two preferences in common.

c) Find \( P(\text{at least 2}) \)

d) Find \( P(\text{less than 2}) \)

Example 7: Use the frequency table at the right to find the probability that a student chosen at random will be 18 to 25 years old.

<table>
<thead>
<tr>
<th>Student age</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>18-25</td>
<td>5</td>
</tr>
<tr>
<td>26-32</td>
<td>15</td>
</tr>
<tr>
<td>33-39</td>
<td>10</td>
</tr>
</tbody>
</table>
**Probability Rules:**

**Range of Probability:** The Probability of an Event must be between 0 and 1, inclusive.

**The Sum Rule:** The probabilities of all outcomes in a sample space must have a sum of 1, or 100%

**The Complement Rule:** The probability of all the outcomes that are not in event E are denoted E'.

\[ P(E) + P(E') = 1 \quad \text{and} \quad 1 - P(E) = P(E') \]

**Example 8:**
The probability that a student who has not received a flu shot will get the flu is 45%

What is the complement of “will get the flu”?

What is the probability of the complement?

**Example 9:** You collect data about favorite color from your senior class. Which of the following probabilities are possible?

- **a)** Red = 0.25 Blue = 0.25 Purple = 0.20 Green = 0.30
- **b)** Red = 0.35 Blue = 0.35 Purple = 0.20 Green = 0.30
- **c)** Red = 0.10 Blue = 0.40 Purple = 0.70 Green = -0.20
- **d)** Red = 1.0 Blue = 0.0 Purple = 0.0 Green = 0.0

**Probability Related to Statistics**

Statistics and Probability are related, but ______________________ mathematical fields.

Probability makes statements about what will occur when samples are drawn from ______________ populations.

Statistics describes to collect data how inferences are made about ______________ populations.
3.2 Conditional Probability and the Multiplication Rule

Suppose you are throwing two fair dice.

a) $P(\text{sum of 3})$

b) What if we throw the white die BEFORE the black die and we get a “1”? Now what is the probability of getting a sum of 3 on the two dice?

This is called **CONDITIONAL PROBABILITY**.

**Conditional Probability**

 Conditional probability is the probability that a dependent event will occur given that another event has occurred.

Notation: $P(A \mid B) = P(\text{A will occur given that B has occurred})$

**Example 1:** All 7000 students at a college were surveyed about their residency status (live on campus, live off campus but in town, or commute). The students were grouped by gender. The results are showing in the following contingency table.

<table>
<thead>
<tr>
<th></th>
<th>On Campus</th>
<th>Off Campus</th>
<th>Commute</th>
<th>Row Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Male Students</strong></td>
<td>1700</td>
<td>1300</td>
<td>600</td>
<td>3600</td>
</tr>
<tr>
<td><strong>Female Students</strong></td>
<td>1800</td>
<td>1200</td>
<td>400</td>
<td>3400</td>
</tr>
<tr>
<td><strong>Column Total</strong></td>
<td>3500</td>
<td>2500</td>
<td>1000</td>
<td>7000</td>
</tr>
</tbody>
</table>

Suppose a student is selected at random from these 7000 students. Consider these events:

- $M = \text{male student}$
- $F = \text{female student}$
- $C = \text{on campus}$
- $O = \text{off campus}$
- $T = \text{commute}$

a) Find $P(C)$

b) Find $P(M)$
Suppose a student is selected at random from these 7000 students. Consider these events:

\[ M = \text{male student} \quad F = \text{female student} \quad C = \text{on campus} \quad O = \text{off campus} \quad T = \text{commute} \]

- **c) Find** \( P(C|M) \)
- **d) Find** \( P(M|C) \)
- **e) Find** \( P(O|F) \)
- **f) Find** \( P(T|M) \)
- **g) Find** \( P(F|C) \)

**MULTIPLE EVENTS; Independent and Dependent Events**

What if we are rolling a die THREE times or drawing TWO cards, one right after the other...

Two events are **independent** if the occurrence or nonoccurrence of one does not ____ the probability that the other will occur.

**INDEPENDENCE and the special multiplication rule.**

TWO EVENTS E AND F ARE INDEPENDENT OF EACH OTHER IF THE OCCURRENCE OF ONE HAS NO INFLUENCE ON THE PROBABILITY OF THE OTHER. FOR INSTANCE, THE ROLL OF ONE DIE HAS NO EFFECT ON THE ROLL OF ANOTHER (UNLESS THEY'RE GLUED TOGETHER, MAGNETIC, ETC.).

Test for Independence:

A and B are independent if:

\[ P(A) = P(A|B) \]

**Example 2:** You roll a standard die and you flip a coin. Find the probabilities.

- **a.** \( P(\text{even number and heads}) \)
- **b.** \( P(4 \text{ and T}) \)
Example 3: Select a card from a deck of cards. Replace the card. Select a second card.

a. \( P(\text{Queen and Ace}) \)  
b. \( P(\text{2 Queens}) \)

Two events are **dependent** if the probability of one event depends upon the occurrence of the other event.

In order to calculate the probability of a dependent event, you need to use Conditional Probability”

**EXAMPLES:**

**Example 4:**
Suppose you are going to throw two fair dice. What is the probability of getting a 3 on each die?

**Example 5:**
Suppose you are drawing two cards without replacement from a well-shuffled deck of 52 cards. What is the probability of getting two hearts?

**Example 6:**
John and Lisa are going to take the driver’s license test. Based on their skills, the probability that John will pass the test is 0.7 and the probability that Lisa will pass is 0.8. Assuming that the test result of one of them has no effect on the result of the other...

a) Are the events *John will pass* and *Lisa will pass* dependent or independent?

b) What is the probability that both will pass?

**Example 7:** You consistently hit 80% of your free throw shots when you practice. What is the probability of hitting the next three shots?
Example 8:
In a jury selection pool, 65% of the people are female. Of these females, one out of four works in a health field.

a) Find the probability that a randomly selected female is a female who works in a health field.

b) Find the probability that a randomly selected female does not work in a health field.

Example 9: You pack a cooler for the beach and stock it with soda cans. You include a 12 pack of cola, a 12 pack of diet cola, and a 6 pack of red bull energy drink. You randomly grab 3 cans in a row and hand them to friends. What is the probability that you:

a) select 3 red bull energy drinks

b) select a cola, a diet cola, and a red bull (in that order)

Example 10 Your teacher is trying to demonstrate a probability concept and is running out of ideas. She creates three bags and fills them with 20 scrabble tiles.

First Bag:

a) You draw 3 tiles out of the first bag, one after the other, without replacement.

P(3 vowels) =

Second Bag:

b) You draw one tile out of each bag.

P(3 vowels) =

c) You draw two tiles out of the second bag, one after the other, without replacement

P(not drawing an E at all) =
Example 11:
You are going to spin the colored spinner at the right.
The spinner has 2 blue regions, 3 yellow regions, and 1 each of yellow, orange, and green.

1. What is the probability of the spinner landing on blue?

2. What is the probability of the spinner landing on any color except blue?

You decide to spin the spinner four times.

3. What is the probability of getting blue each time?

4. What is the probability of not getting blue until your fourth spin?

5. What is the probability of getting at least one blue?

6. What is the probability of getting a least one red?

7. What is the probability of getting “not all blues”?

“At LEAST ONE”
The complement of “At Least One” is:

____________________

These two probabilities should add up to 100%.
3.3 The Addition Rule: What to do with “OR”

A few basic ideas:

\[ E \text{ and } F : \text{ the event } E \text{ and the event } F \text{ both occur.} \]

\[ E \text{ or } F : \text{ the event } E \text{ or the event } F \text{ occurs (or both do).} \]

\[ \text{not } E : \text{ the event } E \text{ does not occur.} \]

Example 1: You roll a standard die one time.

\[ \begin{array}{cccccc}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} \\
\end{array} \]

a) \( P(\text{even and prime}) \)  

b) \( P(\text{even or prime}) \)

c) \( P(\text{not prime}) \)  

d) \( P(\text{even and less than 5}) \)

e) \( P(\text{even or less than 5}) \)  

f) \( P(\text{less than 2 or more than 5}) \)

g) \( P(\text{less than 2 and more than 5}) \)

CAUTION: The word “AND” requires context interpretation.

**PROBABILITY SITUATION 1: Compound Event**
ONE THING is happening or multiple things are happening SIMULTANEOUSLY.

You draw 1 card.

event: (Queen and Heart) = 1 outcome, the Queen of Hearts

THINK: OVERLAP  
\( P(\text{Queen and Heart}) = \frac{1}{52} \)

**PROBABILITY SITUATION 2: Multiple Event**
SEPARATE THINGS are happening or one thing is happening REPITITIOUSLY.

You draw 2 cards, one after another with replacement

event: (Queen and Heart) \( \rightarrow \) means (Queen 1\textsuperscript{st}, Heart 2\textsuperscript{nd})

THINK: MULTIPLE EVENTS  
\( P(\text{Queen and Heart}) = \frac{4}{52} \times \frac{13}{52} \)
Mutually Exclusive Events

Two events are mutually exclusive or _________________ if there are no elements in common to both sets.
In other words:
\[ P(A \text{ and } B \text{ occurring at the same time}) = \]

Addition Rules

If \( A \) and \( B \) are mutually exclusive, then
\[ P(A \text{ or } B) = P(A) + P(B). \]

If \( A \) and \( B \) are not mutually exclusive, then
\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B). \]

Example: \( P(\text{King or Queen}) \) \( P(\text{King or Heart}) \)

Example 2: Suppose you are drawing one card from a well-shuffled deck of 52 cards. *See Cards and Dice Handout*

a) \( P(\text{Ace and Clubs}) \)  

b) \( P(\text{Ace or Clubs}) \)

d) \( P(\text{not Ace}) \)  

e) \( P(\text{hearts or spades}) \)

f) \( P(\text{King or Spades}) \)  

g) \( P(\text{Ace or Black Card}) \)

Example 3: Suppose you are throwing two fair dice.

a) What is the probability that you will roll a sum less than four or bigger than 10?

b) What is the probability that you will roll doubles or a sum bigger than 10?
Example 4: Use the contingency table to find the indicated probabilities. Students at a local school were asked to report their favorite leisure activities.

<table>
<thead>
<tr>
<th></th>
<th>Sports</th>
<th>Hiking</th>
<th>Reading</th>
<th>Phoning</th>
<th>Shopping</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Female</strong></td>
<td>39</td>
<td>48</td>
<td>85</td>
<td>62</td>
<td>71</td>
<td>29</td>
</tr>
<tr>
<td><strong>Male</strong></td>
<td>67</td>
<td>58</td>
<td>76</td>
<td>54</td>
<td>68</td>
<td>39</td>
</tr>
</tbody>
</table>

a) $P(\text{Female and Reading})$  

b) $\text{Female or Reading}$

c) $P(\text{Sports or Hiking})$  

d) $P(\text{Sports and Hiking})$

e) $P(\text{Sports} \mid \text{Male})$  

f) $P(\text{Female} \mid \text{Sports})$

3.4 Additional Topics in Probability and Counting – Permutations and Combinations

Recall the Fundamental Counting Principle from Lesson 3.1:

Example 1: You toss a coin three times. How many possible outcomes are there?

Example 2: There are three reference books, five cookbooks, and four novels on the shelf. Jeff is going to pick one book of each kind. How many different sets are possible?
Factorials and Permutations

Example 3:
In how many ways can five books be arranged on a shelf?

Example 4:
In Sudoku, the digits 1 to 9 must be arranged in each row, column, and 3x3 cage of a 9x9 grid. In how many ways can you arrange the numbers on the top row of the grid?

Example 5:
There are 25 students in Mrs. Leahy’s class. How many ways can she arrange the students in the front row of 8 chairs for her seating chart. (Assume no empty seats)

Example 6:
For a group of 10 people, how many ways can four be seated on the log ride at an amusement park?

Combinations

A combination of “n” objects taken “r” at a time without regard to order is:

\[ {}_n C_r = \frac{n!}{(n-r)!r!} \]

where \( r \leq n \)

*THERE IS A BUTTON ON YOUR CALCULATOR FOR THIS!*

Example 7: The Indiana Department of Transportation is planning on changing every stop light in Hendricks County to a roundabout. Sixteen companies bid for the jobs. The state plans on choosing four companies to hire to complete the work, which will last for the next 10 years. In how many ways can they choose the 4 companies?
Example 8: Your cell phone and laptop are broken. You discover that you will have to read to occupy yourself in SRT. There are eight different books on the shelf at home. How many different groups of three books can be selected from the shelf to bring to school?

Visually: Counting principle, Permutation, Combinations

<table>
<thead>
<tr>
<th>Counting Principle</th>
<th>Permutation</th>
<th>Combination</th>
</tr>
</thead>
</table>

Examples

Example 9: The Bad Wolf Observational Society has 20 members.

a) How many ways could the club select a committee three members to attend GenCon this year?

b) How many ways could the club elect a president, vice president, secretary, and treasurer?

Example 10: You are buying a new car and have the choice of three different wheel types, two different security packages, and five different colors. How many different combinations of one wheel type, one security package, and one color are possible?

Example 11: A raffle is being held among Mrs. Leahy’s 98 Probability and Statistics Students. Each student’s name is written on a slip of paper and then Mrs. Leahy will draw out names. How many ways are there to give away:

a) Three identical Starbucks gift cards

b) Three prizes: $100 Amazon giftcard, a “Free Drink” coupon from Taco Bell, an autographed photo of Mrs. Leahy.
IN CLASS REVIEW:
Examples:

1. The cup on my desk contains 3 red pens, 4 blue pens, 2 black pens, 2 green pens, and 1 orange pen. If I randomly select one pen to grade with, find the following probabilities.
   a) \( P(\text{green}) \)  
   b) \( P(\text{not red}) \)  
   c) \( P(\text{purple}) \)

2. You roll a fair die. What is the probability of rolling a number of at least a 4 on a single throw?

3. You roll two dice: one is black and one is white
   What is the probability of rolling an even number on the black die and a 5 on the white die?

4. You write the 26 letters of the English Alphabet on individual cards and shuffle them.
   Vowels: AEIOU  
   Consonants: BCDFGHJKLMNPQRSTVWXYZ
   a) You draw two cards from the deck, but replace the first card before drawing the second card.
      What is the probability that the first card is a vowel and the second card is “W, X, Y, or Z”?
   b) You draw two cards from the deck, but do not replace the first card before drawing the second card.
      What is the probability that the first card is a vowel and the second card is “W, X, Y, or Z”?
   c) You draw three cards from the deck, without replacement. What is the probability that all three cards are vowels?

5. In your stats class, you learn that 30% of students got an A on the last test. Of the students who got an A, 85% of them had no missing homework assignments. What is the probability that a student selected at random from the class got an A on the last test and had no missing homework assignments?

6. There is a 25% chance of getting stopped by the CSX train at SR 267 between 7:45AM and 8:00AM. What is the probability of getting stopped three days in a row?
7. Adults and Children were surveyed and asked to state if they preferred chocolate or vanilla ice cream. The results are shown in the table.

If a survey participant is selected at random, what are the probabilities of the following:

a) \( P(\text{Vanilla}) \)

b) \( P(\text{Chocolate and Child}) \)

c) \( P(\text{Child}) \)

d) \( P(\text{Adult} | \text{Chocolate}) \)

e) \( P(\text{Vanilla} | \text{Adult}) \)

f) \( P(\text{Vanilla or Child}) \)

<table>
<thead>
<tr>
<th></th>
<th>Adult</th>
<th>Child</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vanilla</td>
<td>52</td>
<td>26</td>
<td>78</td>
</tr>
<tr>
<td>Chocolate</td>
<td>41</td>
<td>105</td>
<td>146</td>
</tr>
<tr>
<td>Total</td>
<td>93</td>
<td>131</td>
<td>224</td>
</tr>
</tbody>
</table>

8. A nail salon has 14 new colors of polish. How many ways can 5 different polishes be chosen to put on a display for the front window?

9. The band boosters sell 75 raffle tickets. First place wins $100, Second place $25, and 3\(^{rd}\) place wins a coupon for Orange Leaf. How many ways can a 1\(^{st}\), 2\(^{nd}\), and 3\(^{rd}\) place winner be chosen?

10. Mrs. Leahy has a collection of 10 corny jokes, 8 random anecdotes, and 5 bizarre movie references. How many ways can she choose 1 joke, 1 anecdote, and 1 reference to tell her long suffering math students?

11. **True or False:** The Probability of an event is negative if the event is very unlikely.

12. **True or False:** The complement of “The pen is red” is “The pen is blue.”