Area is only one of the many applications of the definite integral. We can also use it to find the volume of a three-dimensional solid. The solids which you will focus on in this unit are those with cross sections that are all similar.

A 2-dimensional figure ..........which, when rotated about the x-axis becomes a.......3-dimensional figure

Recall: The formula for the volume of a cylinder is \( V = \pi r^2 h \). A disk is basically a very short cylinder where the height, \( h \) is actually the \( w \) value. The radius retains its actual meaning. The \( \pi \) pops out in front and you get the following formula below:

**The Disk Method**

To find the volume of a solid of revolution with the **disk method**, use one of the following:

### Horizontal Axis of Revolution

Volume \( V = \pi \int_b^a [R(x)]^2 \, dx \)

### Vertical Axis of Revolution

Volume \( V = \pi \int_c^d [R(y)]^2 \, dy \)

**Example 1: Using the Disk Method.**

Find the volume of the solid formed by revolving the region bounded by the graphs of \( f(x) = \sqrt{\sin x} \) and the x-axis about the x-axis \( (0 \leq x \leq \pi) \).

Sketch the graph and draw a representative rectangle.

\[
V = \pi \int_0^\pi \left(\sqrt{\sin x}\right)^2 \, dx
\]

\[
= \pi \int_0^\pi \sin x \, dx
\]

\[
= \left. \pi (-\cos x) \right|_0^\pi
\]

\[
= \pi (-(1) - (1)) = 2\pi
\]

\[
= \pi (\cos 0 - (-\cos 0)) = 2\pi
\]

Find the volume of the solid formed by revolving the region bounded by the graphs of \( f(x) = 2 - x^2 \) and \( g(x) = 1 \) about the line \( y = 1 \).

Sketch the graph and draw a representative rectangle.

\[
\sqrt{\pi} \int_{-1}^{1} ((2-x^2) - 1)^2 \, dx
\]

\[
= \pi \int_{-1}^{1} (1-x^2)^2 \, dx
\]

\[
= \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, dx
\]

\[
= \pi \left[ x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1}
\]

\[
= \pi \left[ (1 - \frac{2}{3} + \frac{1}{5}) - (\frac{2}{3} - \frac{1}{5}) \right]
\]

\[
= \pi \left[ \frac{8}{15} - \left( -\frac{8}{15} \right) \right]
\]

\[
= \frac{16\pi}{15}
\]

The Washer Method
To find the volume of a solid of revolution with the washer method, use one of the following:

Horizontal Axis of Revolution

Volume \( V = \pi \int_{a}^{b} \left( [R(x)]^2 - [r(x)]^2 \right) \, dx \)

Vertical Axis of Revolution

Volume \( V = \pi \int_{c}^{d} \left( [R(y)]^2 - [r(y)]^2 \right) \, dy \)

Example 3: Using the Washer Method.

Find the volume of the solid formed by revolving the region bounded by the graphs of \( f(x) = \sqrt{x} \) and \( y = x^2 \) about the x-axis.

Sketch the graph and draw a representative rectangle.

\[
V = \pi \int_{0}^{1} \left( [R(x)]^2 - [c(x)]^2 \right) \, dx
\]

\[
= \pi \int_{0}^{1} \left( (\sqrt{x})^2 - (x^2 - 0)^2 \right) \, dx
\]

\[
= \pi \int_{0}^{1} \left( x - x^4 \right) \, dx
\]

\[
= \pi \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_{0}^{1}
\]

\[
= \pi \left( \frac{1}{2} - \frac{1}{5} \right) - (0 - 0)
\]

\[
= \pi \left( \frac{3}{10} \right)
\]

\[
= \frac{3\pi}{10}
\]
Example 4: Integrating with Respect to y, Two Integral Case.
Find the volume of the solid formed by revolving the region bounded by the graphs of
\( y = x^2 + 1 \), \( y = 0 \), \( x = 0 \), and \( x = 1 \) about the y-axis.
Sketch the graph and draw a representative rectangle.

\[
\begin{align*}
\sqrt{\pi} & = \pi \int_0^1 (1 - 0)^2 \, dy + \pi \int_1^2 \left[ (1 - 0)^2 - (\sqrt{y} - 1 - 0)^2 \right] \, dy \\
& = \pi \int_0^1 dy + \pi \int_1^2 (2 - y) \, dy \\
& = \pi \left[ y \right]_0^1 + \pi \left[ 2y - \frac{y^2}{2} \right]_1^2 \\
& = \pi (1 - 0) + \pi \left( (4 - 3) - (2 - \frac{1}{2}) \right) \\
& = \pi + \frac{1}{2} \pi \\
& = \frac{3\pi}{2}
\end{align*}
\]

Think of the above example in some other ways.
What would your integration set-up look like if you were to revolve the region about
(a) the line \( x = -1 \)?

\[
\begin{align*}
V &= \pi \int_0^1 \left( (1 - 1)^2 - (0 - 1)^2 \right) \, dy \\
& + \pi \int_2^1 \left( \left( \sqrt{y} - 1 \right)^2 - (0 - 1)^2 \right) \, dy
\end{align*}
\]

(b) the line \( y = -2 \)?

\[
\begin{align*}
V &= \pi \int_0^1 \left( (x^2 + 1 - 2)^2 - (0 - 2)^2 \right) \, dx \\
& + \pi \int_1^2 \left( \left( \sqrt{y} - 1 \right)^2 - (2 - 1)^2 \right) \, dy
\end{align*}
\]

(c) the line \( x = 2 \)?

\[
\begin{align*}
V &= \pi \int_0^1 \left( (2 - 0)^2 - (2 - 1)^2 \right) \, dy \\
& + \pi \int_1^2 \left( \left( \sqrt{y} - 1 \right)^2 - (2 - 1)^2 \right) \, dy
\end{align*}
\]

Advice

1. Draw a representative rectangle that begins from the axis of revolution to the farthest boundary.
   (This distance will represent R)
2. Draw a representative rectangle that begins from the axis of revolution to the nearest boundary.
   (This distance will represent r)
### Solids With Known Cross Sections

#### Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis.
   \[
   \text{Volume} = \int_a^b A(x) \, dx
   \]

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis.
   \[
   \text{Volume} = \int_c^d A(y) \, dy
   \]

---

#### Example 5: Triangular Cross Sections.

Find the volume of the solid in which the base is the region bounded by the graphs of 

\[ f(x) = 1 - \frac{x}{2}, \quad g(x) = -1 + \frac{x}{2}, \quad \text{and} \quad x = 0. \]

a. The cross sections perpendicular to the x-axis are squares.

\[
V = \int_0^2 \left(2-x\right)^2 \, dx
= \int_0^2 \left(4 - 4x + x^2\right) \, dx
= 4x - 2x^2 + \frac{x^3}{3} \bigg|_0^2
= \left(8 - 8 + \frac{8}{3}\right) - (0) = \frac{8}{3}
\]

b. The cross sections perpendicular to the x-axis are semi-circles.

\[
V = \frac{1}{8} \pi \int_0^2 \left(2-x\right)^2 \, dx
= \frac{1}{8} \pi \int_0^2 \left(4 - 4x + x^2\right) \, dx
= \frac{1}{8} \pi \left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2
= \frac{1}{8} \pi \left(\frac{8}{3}\right) = \frac{\pi}{3}
\]

c. The cross sections perpendicular to the x-axis are equilateral triangles.

\[
V = \frac{\sqrt{3}}{4} \int_0^2 \left(2-x\right)^2 \, dx
= \frac{\sqrt{3}}{4} \left(\frac{8}{3}\right) = \frac{2\sqrt{3}}{3}
\]