In this section we will look at a different approach in finding the volume of a solid of revolution. Instead of using disks and washers (disks with holes), we will use cylindrical tubes, or shells.

I will call this tube a “shell.” Therefore it is true that its volume can be found this way:

\[ V_{\text{Shell}} = V_{\text{Outer Cylinder}} - V_{\text{Inner Cylinder}} \]

To compute this we would see that

\[
V_{\text{Shell}} = \pi \left( p + \frac{w}{2} \right)^2 h - \pi \left( p - \frac{w}{2} \right)^2 h
\]

\[
= \pi \left( p^2 + pw + \frac{w^2}{4} \right) h - \pi \left( p^2 - pw + \frac{w^2}{4} \right) h
\]

\[
= \pi hp^2 + \pi hpw + \pi h\frac{w^2}{4} - \pi hp^2 + \pi hpw - \pi h\frac{w^2}{4}
\]

\[
= 2\pi hpw
\]

= 2\pi(average radius)(height)(thickness)

**The Shell Method**

To find the volume of a solid of revolution with the **shell method**, use one of the following:

**Vertical Axis of Revolution**

Volume \( V = \pi \int_a^b p(x)h(x)dx \)

**Horizontal Axis of Revolution**

Volume \( V = \pi \int_c^d p(y)h(y)dy \)

* Whenever the axis of revolution is the y-axis, will use \( p(x) = x \). With all other vertical axes you can find \( p(x) \) by using a “right - left” technique.

* Whenever the axis of revolution is the x-axis, you you will use \( p(y) = y \). With all other horizontal axes you can find \( p(y) \) by using a “top - bottom” technique.
Example 1: Using the Shell Method.

Find the volume of the solid formed (in quadrant 1 only) by revolving the region bounded by the graphs of $f(x) = x - x^3$ and the x-axis about the y-axis.

Sketch the graph and draw a representative rectangles where needed.

$$V = 2\pi \int_0^1 (x-0)(x-x^3 - 0) \, dx$$

$$= 2\pi \int_0^1 (x^2 - x^4) \, dx$$

$$= 2\pi \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= 2\pi \left[ \left( \frac{1}{3} - \frac{1}{5} \right) - (0-0) \right]$$

$$= \left[ \frac{4\pi}{15} \right]$$

Example 2: Using the Shell Method.

Find the volume of the solid formed by revolving the region bounded by the graphs of $x = e^{-y^2}$, $0 \leq y \leq 1$ and the y-axis about the x-axis.

Sketch the graph and draw a representative rectangles where needed.

$$V = 2\pi \int_0^1 (y-0)(e^{-y^2} - 0) \, dy$$

$$= 2\pi \int_0^1 ye^{-y^2} \, dy$$

$$= 2\pi \left[ -\frac{1}{2} e^{-y^2} \right]_0^1$$

$$= -\pi \left[ e^{-y^2} \right]_0^1$$

$$= -\pi \left( e^{-1} - 1 \right) \approx 1.986$$

Example 3: Shell Method Preferable.

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

Sketch the graph and draw a representative rectangles where needed.

$$V = 2\pi \int_0^1 (x-0)(x^2 + 1 - 0) \, dx$$

$$= 2\pi \int_0^1 (x^3 + x) \, dx$$

$$= 2\pi \left[ \frac{x^4}{4} + \frac{x^2}{2} \right]_0^1$$

$$= 2\pi \left[ \left( \frac{1}{4} + \frac{1}{2} \right) - (0+0) \right]$$

$$= 2\pi \left( \frac{3}{4} \right) = \left[ \frac{3\pi}{2} \right]$$
Example 4: Shell Method Necessary.
Find the volume of the solid formed by revolving the region bounded by the graphs of
\( y = x^3 + x + 1 \), \( y = 1 \) and \( x = 1 \) about the line \( x = 2 \).
Sketch the graph and draw a representative rectangles where needed.

\[
V = 2\pi \int_{0}^{1} (2-x)(x^3+x+1-1) \, dx \\
= 2\pi \int_{0}^{1} (2-x)(x^3+x) \, dx \\
= 2\pi \int_{0}^{1} (-x^4 + 2x^3 - x^2 + 2x) \, dx \\
= 2\pi \left[ -\frac{x^5}{5} + \frac{2x^4}{4} - \frac{x^3}{3} + x^2 \right]_{0}^{1} \\
= 2\pi \left[ \left( -\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1 \right) - (0) \right] \\
= 2\pi \left( \frac{29\pi}{30} \right) \\
= \frac{29\pi^2}{15}
\]

Advice
1. Draw an arbitrary representative rectangle that is PARALLEL to the axis of revolution.
2. To find your radius, \( p(x) \) or \( p(y) \), draw a representative line that begins from the axis of revolution and extend it to the middle of the arbitrarily drawn representative rectangle drawn in Step 1.
   Calculate the length of that line by the “top - bottom” or “right - left” technique.
3. Calculate the length of that representative rectangle, \( h(x) \) or \( h(y) \), by the “top - bottom” or “right - left” technique.