Let $f$ be a function having derivatives for all orders of real numbers. The fourth-degree Taylor polynomial for $f$ about $x = 2$ is given by

$$ P(x) = 20 - \frac{11}{2}(x-2)^2 + \frac{5}{3}(x-2)^3 + k(x-2)^4, $$

where $k \neq 0$.

a. Find $f(2)$, $f''(2)$, and $f'''(2)$. (3)

\[
\begin{align*}
  f(2) &= 20 \\
  f''(2) &= \frac{-11}{2} \cdot 2! = -11 \\
  f'''(2) &= \frac{5}{3} \cdot 3! = 10
\end{align*}
\]

b. Determine whether $f$ has a relative minimum, relative maximum, or neither at $x = 2$. Justify your answer. (3)

\[
\begin{align*}
  f'(x) &\approx P'(x) = -11(x-2) + 5(x-2)^2 + 4k(x-2)^3 \\
  f'(2) &\approx P'(2) = 0 \\
  f''(2) &\approx P''(2) = -11
\end{align*}
\]

**By the Second Derivative Test, $f(x)$ has a relative maximum at $x = 2$.**

c. If $f^{(4)}(2) = \frac{3}{2}$, show that $k$ must equal $\frac{1}{16}$. (1)

\[
\frac{3}{2} = \frac{3}{2} \cdot \frac{1}{24} = \frac{3}{48} = \frac{1}{16}
\]

d. Using the value of $k$ from part c, use $P(x)$ to approximate $f(4)$. Is there enough information to determine whether $f$ has a critical point at $x = 4$? If not, explain why. If so, determine whether $f(4)$ is a relative minimum, relative maximum or neither. Justify your answer. (2)

\[
\begin{align*}
  f(4) &\approx 20 - \frac{11}{2}(4-2)^2 + \frac{5}{3}(4-2)^3 + \frac{1}{16}(4-2)^4 \\
  &\approx 20 - 22 + \frac{40}{3} + 1 = \frac{37}{3} \\
  f'(4) &\approx P'(4) = -11(4-2) + 5(4-2)^2 + \frac{1}{4}(4-2)^3 \\
  &\approx -22 + 20 + 2 = 0
\end{align*}
\]

Not enough information at $x = 4$ to determine extrema. Stu Schwartz
Let $g$ be a function having derivatives for all orders for $x \geq 0$. Selected values of $g$ and its first three derivatives are indicated in the table below. The function $g$ and the first two derivatives are increasing on the interval $0 \leq x \leq 2$ while the third derivative is decreasing on the interval $0 \leq x \leq 2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
<th>$g''(x)$</th>
<th>$g'''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>11</td>
<td>$\frac{723}{10}$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>9</td>
<td>$\frac{53}{3}$</td>
<td>$\frac{99}{4}$</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>$\frac{21}{2}$</td>
<td>$\frac{214}{5}$</td>
<td>8</td>
</tr>
</tbody>
</table>

a. Write the first-degree Taylor polynomial for $g$ about $x = 1$ and use it to approximate $g(1.1)$. Is this approximation greater than or less than $g(1.1)$? Explain. (4)

\[ g(x) \approx 5 + 9(x - 1) \]
\[ g(1.1) \approx 5 + 9(0.1) \]
\[ \approx 5.9 \]

\[ \text{Since } g''(1) > 0 \]
\[ g(x) \text{ is concave up at } x = 1, \text{ near } x = 1.1 \]
\[ \therefore \text{The approximation is less than } g(1.1). \]

b. Write the second-degree Taylor polynomial for $g$ about $x = 1$ and use it to approximate $g(1.1)$. (2)

\[ g(x) \approx P_3(x) = 5 + 9(x - 1) + \frac{53}{6} (x - 1)^2 \]
\[ \approx 5 + 9(0.1) + \frac{53}{6} (0.1)^2 \]
\[ \approx 5.988 \]

c. Use the Lagrange error bound to show that the second-degree Taylor polynomial for $g$ about $x = 1$ approximates $g(1.1)$ with error less than 0.01. (3)

\[ |R_3| = \left| \frac{99}{4 \cdot 3!} (1.1 - 1)^3 \right| = 0.004125 \]
The Taylor series about \( x = 0 \) for a certain function \( f \) converges to \( f(x) \) for all \( x \) in the interval of convergence. The \( n \)th derivative at \( x = 0 \) is given by

\[
f^{(n)}(0) = \frac{(-1)^n(n+2)!}{4^{n-1}n^2} \quad \text{for } n \geq 2
\]

The graph of \( f \) has a horizontal tangent at \( x = 0 \) and \( f(0) = \frac{-3}{2} \).

a. Determine whether \( f \) has a relative maximum, relative minimum, or neither at \( x = 0 \). Justify your answer. (2)

\[
f'(0) = 0 \quad \text{(due to horizontal tangent at } x = 0)\]

\[
f''(0) = \frac{(-1)^2(4)!}{4! \cdot 2} = \frac{3}{2} > 0
\]

By The Second Derivative Test, \( f(x) \) has a relative minimum at \( x = 0 \).

b. Write the third-degree Taylor polynomial for \( f \) about \( x = 0 \). (3)

\[
f^{(0)}(0) = -\frac{3}{2}
\]

\[
f^{(1)}(0) = 0
\]

\[
f^{(2)}(0) = \frac{3}{2}
\]

\[
f^{(3)}(0) = \frac{(-1)^3(5!)}{4^2 \cdot 3^2} = -\frac{120}{144} = -\frac{5}{6}
\]

\[
P_3(x) = -\frac{3}{2} + \frac{3}{2 \cdot 2!}x^2 - \frac{5}{6 \cdot 3!}x^3
\]

or

\[
-\frac{3}{2} + \frac{3}{4}x^2 - \frac{5}{36}x^3
\]

c. Find the radius of convergence of the Taylor Series for \( f \) about \( x = 0 \). Show the work that leads to your answer. (4)

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+3)! \cdot \chi^{n+1}}{4^n(n+1)^2 \cdot (n+1)!} \cdot \frac{4^{n-1} \cdot n^2 \cdot n!}{(-1)^n(n+2)! \cdot \chi^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(-1)(n+3) \cdot n^2 \cdot \chi}{(n+1)(n+1)^2 \cdot 4} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{n^2(n+3)}{(n+1)(n+1)^2 \cdot 4} \cdot \lim_{n \to \infty} \left| \frac{\chi}{\frac{4}{n}} \right| = \left| \frac{\chi}{\frac{4}{4}} \right| \text{ which must be } < 1
\]

\[-1 < \chi < 1\]

\[-4 < \chi < 4\]  \( R = 4 \)
Let $f$ be the function given by $f(x) = e^{\sqrt{x}} + e^{-\sqrt{x}}$

a. Show that the first three nonzero terms of the Taylor series for $f$ about $x = 0$ is $2 + x + \frac{x^2}{12}$. Find the general term as well. (3)

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^{\sqrt{x}} = 1 + \sqrt{x} + \frac{x}{2} + \frac{(\sqrt{x})^3}{3!} + \frac{x}{4!} + \cdots$

$e^{-\sqrt{x}} = 1 - \sqrt{x} + \frac{x}{2} - \frac{(\sqrt{x})^3}{3!} + \frac{x^2}{4!} + \cdots$

$e^{\sqrt{x}} + e^{-\sqrt{x}} = 2 + \frac{2x}{2} + \frac{2x^2}{4!}$

$= 2 + x + \frac{x^2}{12}$

b. Use your answer to part a) to find $\lim_{x \to 0^+} \frac{2 + x - f(x)}{x^2}$ (2)

$\lim_{x \to 0^+} \frac{2 + x - (2 + x + \frac{x^2}{2} + \cdots + \frac{2x^2}{12} + \cdots)}{x^2} = \lim_{x \to 0^+} \left( \frac{\frac{2x^n}{(2n)!}}{x^2} \right) = \left[ \frac{-1}{12} \right]$ (2)

c. Write the first four nonzero terms of the Taylor series for $\int_0^x f(-t) \, dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} f(-t) \, dt$. (3)

$f(-t) = \frac{2t}{2!} + \frac{2t^2}{4!} - \frac{2t^3}{6!} + \cdots$

$\int_0^x f(-t) \, dt = \int_0^x \left( \frac{2t}{2!} + \frac{2t^2}{4!} - \frac{2t^3}{6!} + \cdots \right) \, dt = 2x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{144} + \cdots$

$\int_0^{1/2} f(-t) \, dt \approx 2 \left( \frac{1}{2} \right) - \frac{(\frac{1}{2})^2}{2} = 1 - \frac{1}{8} = \left[ \frac{7}{8} \right]$

d. Find the maximum error in the estimate in part c. (1)

The first neglected term is $\frac{x^3}{36}$

$\left( \frac{1}{2} \right)^3 = \frac{1}{8} \cdot \frac{1}{36} = \left[ \frac{1}{288} \right]$

$\text{Error} \leq \left| \frac{x^3}{36} \right|_{x=\frac{1}{2}}$