AP CALCULUS BC	LECTURE NOTES	MR. RECORD
Section Number: **9.2**	**Topics: Series and Convergence** - Infinite Series - Geometric Series	**Day:** 1 of 2

Infinite Series

Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Sequence of Partial Sums:
$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

DEFINITION OF CONVERGENT AND DIVERGENT SERIES

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is given by

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ where $S = \sum_{n=1}^{\infty} a_n$

If $\{S_n\}$ diverges, then the series **diverges.**

Example 1: Convergent and Divergent Series.

Find the sequence of partial sums $S_1, S_2, S_3, S_4,$ and S_5 for each of the following series.

a. $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

Partial Sums

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} \frac{(\ln 2)\, 2^n}{(\ln 2)\cdot 2^n} = 1$

$\boxed{\text{the series converges to } 1.}$

b. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$

$S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$

$S_n = \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1$

$\boxed{\text{the series converges to } 1}$

c. $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$

$S_1 = 1$

$S_2 = 2$

$S_3 = 3$

$S_4 = 4$

$S_n = n$

$\lim_{n\to\infty} n = \infty$

$\boxed{\text{the series diverges}}$

Telescoping series have the form $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots$ therefore $S_n = b_1 - b_{n+1}$

A telescoping series converges if and only if $b_n \to$ a finite number as $n \to \infty$. In which case $S = b_1 - \lim_{n\to\infty} b_{n+1}$

In "b." this was 0

Example 2: Writing a Series in Telescoping

Find the sum of the series $\sum_{n=1}^{\infty}\frac{2}{4n^2-1}$.

Using partial fractions,

$a_n = \frac{2}{4n^2-1} = \frac{2}{(2n-1)(2n+1)} \Rightarrow \frac{A}{2n-1} + \frac{B}{2n+1} \Rightarrow 2 = A(2n+1) + B(2n-1)$

$n = -\frac{1}{2}$ $\quad$ $n = \frac{1}{2}$

$2 = B(-2)$ $\quad$ $2 = 2A$

$B = -1$ $\quad$ $A = 1$

$= \frac{1}{2n-1} - \frac{1}{2n+1}$

So, $S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}$

$\lim_{n\to\infty} 1 - \frac{1}{2n+1} = \boxed{1}$

Geometric Series

Specific Example: $\quad 3+3\ 2+3\ 2^2+3\cdot 2^3+\cdots$

General Form: $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot\ + ar^n + \cdot\ \cdot, \quad a \neq 0$

THEOREM 9.6: CONVERGENCE OF A GEOMETRIC SERIES

1. A geometric series with a ratio r diverges if $|r| \geq 1$. $\quad r \geq 1$ or $r \leq -1$
2. If $0 < |r| < 1$, then the series converges to the sum

or $0 < r < 1$ $\quad$ $-1 < r < 0$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Example 3: Convergent and Divergent Geometric Series

Determine the convergence or divergence of the series. If the series converges, find its sum.

a. $\sum_{n=0}^{\infty}\frac{3}{2^n}$

$= \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3(1) + 3\left(\frac{1}{2}\right)^1 + 3\left(\frac{1}{2}\right)^2 + \dots$

$r = \frac{1}{2}$ $(a=3)$

Series converges, $S = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = \boxed{6}$

b. $\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n$

$= 1 + \left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \dots$

$r = \frac{3}{2}$

$\boxed{\text{Series diverges}}$

THEOREM 9.7: PROPERTIES OF INFINITE SERIES

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B, and c be real numbers.

If $\sum a_n = A$ and $\sum b_n = B$, then the following series converge to the indicated sums

1. $\sum^{\infty} ca_n = cA$ $\qquad$ **2.** $\sum^{\infty}(a_n \pm b_n) = A \pm B$

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Section Number: **9.2**	**Topics: Series and Convergence** - *n*th-Term Test for Divergence	**Day:** 2 of 2

*n*th-Term Test for Divergence

THEOREM 9.8: LIMIT OF THE *n*TH TERM OF A CONVERGENT SERIES

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

The contra-positive of Theorem 9.8 follows.

THEOREM 9.9: *n*TH TERM FOR DIVERGENCE

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 4: Using the *n*th-Term Test for Divergence.
Determine the convergence or divergence of each series.

a. $\sum_{n=0}^{\infty} 2^n$

$\lim_{n\to\infty} 2^n = \infty$

$\boxed{\text{The series diverges}}$

b. $\sum_{n=0}^{\infty} \frac{n!}{2n!+1}$

$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2}$

$\boxed{\text{The series diverges}}$

Specific case

$\frac{5!}{2\cdot 5!+1} = \frac{1\cdot2\cdot3\cdot4\cdot5}{2(1\cdot2\cdot3\cdot4\cdot5)+1} = \frac{120}{241}$

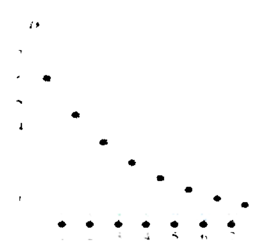
c. $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\boxed{\text{Cannot draw a conclusion from the } n^{th} \text{ term test}}$

* Be sure to look at 83-89 odd

Example 5: Bouncing Ball Problem.
A ball is dropped from a height of 6 feet and begins bouncing as shown in the figure to the right. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



The height of each bounce is three-fourths the height of the previous bounce

D_i = distance ball travels up then down

$D_1 = 6$

$D_2 = \underbrace{6\left(\frac{3}{4}\right)}_{\text{up}} + \underbrace{6\left(\frac{3}{4}\right)}_{\text{down}} = 12\left(\frac{3}{4}\right)$

$D_3 = 6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) + 6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = 12\left(\frac{3}{4}\right)^2$

$D = 6 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + 12\left(\frac{3}{4}\right)^3 + \ldots + = 6 + 12\left(\frac{3}{4}\right)\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = 6 + 9\left(\frac{1}{1-\frac{3}{4}}\right) = 6 + 9(4) = \boxed{42 \text{ ft}}$