AP CALCULUS BC	LECTURE NOTES	MR. RECORD
Section Number: **9.3**	**Topics: The Integral Test and p-Series** - The Integral Test - *p*-Series and The Harmonic Series	**Day:** 1 of 1

The Integral Test

THEOREM 9.10: THE INTEGRAL TEST
If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Example 1: Using the Integral Test.
Confirm whether the Integral Test can be applied to the series. Then use the Integral Test to determine the convergence or divergence of the series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$f(x) = \frac{x}{x^2+1}$ is always positive and continuous for $x \geq 1$

$f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0$ for $x \geq 1$ $\therefore$ $f(x)$ is decreasing for $x \geq 1$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^{\infty} \frac{2x}{x^2+1}\,dx = \frac{1}{2}\lim_{b\to\infty}\int_1^b \frac{2x}{x^2+1}\,dx$$

$u = x^2+1$
$du = 2x\,dx$

$$= \frac{1}{2}\lim_{b\to\infty}\Big[\ln(x^2+1)\Big]_1^b$$

$$= \frac{1}{2}\lim_{b\to\infty}\Big[\ln(b^2+1) - \ln 2\Big] = \infty$$

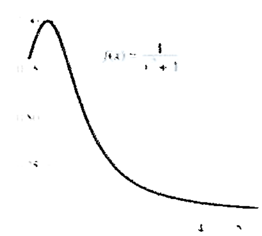
$\boxed{\text{The series diverges}}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$ is always positive and continuous for $x \geq 1$

$f'(x) = -(x^2+1)^{-2}\cdot 2x = \frac{-2x}{(x^2+1)^2} < 0$ for $x \geq 1$

$\therefore$ $f(x)$ is decreasing for $x \geq 1$.

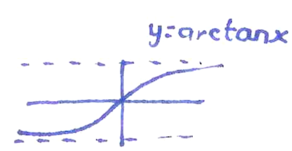


$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \arctan x \Big|_1^b$$

$$= \lim_{b\to\infty} \arctan b - \arctan 1$$

Recall



$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

$\boxed{\text{The series converges}}$

But doesn't necessarily add to $\frac{\pi}{4}$.

p-Series and Harmonic Series

p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

Harmonic Series ($p = 1$)

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots$$

General Harmonic Series

$$\sum_{n=1}^{\infty}\frac{1}{an+b}=\frac{1}{a+b}+\frac{1}{2a+b}+\frac{1}{3a+b}+\cdots$$

Historical Note

Harmonic Series

Pythagoras and his students paid close attention to the development of music as an abstract science. This led to the discovery of the relationship between the tone and the length of the vibrating string. It was observed that the most beautiful musical harmonies corresponded to the simplest ratios of whole numbers. Later mathematicians developed this idea into the harmonic series, where the terms in the harmonic series correspond to the nodes on a vibrating string that produce multiples of the fundamental frequency. For example, $\frac{1}{2}$ is twice the fundamental frequency, $\frac{1}{3}$ is three times the fundamental frequency, and so on.

THEOREM 9.11: CONVERGENCE OF A p-SERIES

The p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

Example 2: Convergent and Divergent p-Series.

Determine the convergence or divergence of the following p-series.

$$\underbrace{1}_{\sqrt[3]{1}}+\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}+\frac{1}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{25}}+\cdots$$

$$=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2}}=\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}.\qquad \boxed{\text{diverges}}$$

Example 3: Testing a Series for Convergence

a. Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

We know $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges but what if the denominator were made bigger? Could we expect it to converge.

$f(x)=\frac{1}{x\ln x}$ is positive for $x\geq 2$ and continuous for $x\geq 2$

$$f'(x)=-(x\ln x)^{-2}(\ln x+1)$$
$$=-\frac{1+\ln x}{x^2(\ln x)^2}<0 \quad \text{for } x\geq 2$$

So we can use the Integral Test

$$\int_2^{\infty}\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\int_2^b\frac{1}{x\ln x}\,dx$$

$u=\ln x \quad du=\frac{1}{x}dx$

$$=\lim_{b\to\infty}\ln(\ln x)\Big|_2^b$$

$$=\lim_{b\to\infty}\left[\ln(\ln b)-\ln(\ln 2)\right]$$

$$=\infty$$

$\boxed{\text{diverges}}$

b. Find the positive values of p for which the series converges.

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$$

p can't be 1

$$\int_2^{\infty}(\ln x)^{-p}\cdot\frac{1}{x}\,dx=\frac{(\ln x)^{-p+1}}{-p+1}\Bigg|_2^{\infty}=\frac{\infty^{-p+1}}{-p+1}-\frac{(\ln 2)^{-p+1}}{-p+1}$$

$\Rightarrow -p+1<0$

$\boxed{p>1}$