5. $\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x}\, dx = \left[-e^{-x}\right]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 9.10

7. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

Let $f(x) = \frac{1}{x^2 + 1}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2 + 1}\, dx = \left[\arctan x\right]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 9.10

9. $\sum_{n=1}^{\infty} \frac{\ln(n + 1)}{n + 1}$

Let $f(x) = \frac{\ln(x + 1)}{x + 1}$.

f is positive, continuous, and decreasing for

$x \geq 2$ because $f'(x) = \frac{1 - \ln(x + 1)}{(x + 1)^2} < 0$ for $x \geq 2$.

$$\int_1^{\infty} \frac{\ln(x + 1)}{x + 1}\, dx = \left[\frac{\left[\ln(x + 1)\right]^2}{2}\right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

15. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Let $f(x) = \frac{\ln x}{x^2}$, $f'(x) = \frac{1 - 2\ln x}{x^3}$.

f is positive, continuous, and decreasing for $x > e^{1/2} \approx 1.6$.

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \left[\frac{-(\ln x + 1)}{x}\right]_1^{\infty} = 1, \text{ converges}$$

So, the series converges by Theorem 9.10.

21. $\sum_{n=1}^{\infty} \frac{4n}{2n^2 + 1}$

Let $f(x) = \frac{4x}{2x^2 + 1}$, $f'(x) = \frac{-4(2x^2 - 1)}{(2x^2 + 1)^2} < 0$

for $x \geq 1$

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{4x}{2x^2 + 1}\,dx = \left[\ln(2x^2 + 1)\right]_1^{\infty} = \infty$, diverges.

So, $\sum_{n=1}^{\infty} \frac{4n}{2n^2 + 1}$ diverges by Theorem 9.10.

29. Let $f(x) = \frac{2 + \sin x}{x}$, $f(n) = a_n$.

The function f is not decreasing for $x \geq 1$.

31. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \frac{1}{x^3}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^{\infty} = \frac{1}{2}$

Converges by Theorem 9.10

33. Let $f(x) = \frac{1}{x^{1/4}}$, $f'(x) = \frac{-1}{4x^{5/4}} < 0$ for $x \geq 1$

f is positive, continuous, and decreasing for $x \geq 1$

$\int_1^{\infty} \frac{1}{x^{1/4}}\,dx = \left[\frac{4x^{3/4}}{3}\right]_1^{\infty} = \infty$, diverges

So, the series diverges by Theorem 9.10.

35. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent p-series with $p = \frac{1}{5} < 1$

37. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent p-series with $p = \frac{1}{2} < 1$

39. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent p-series with $p = \frac{3}{2} > 1$

41. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent p-series with $p = 1.04 > 1$

59. $\sum_{n=1}^{\infty} \frac{n}{(1 + n^2)^p}$

If $p = 1$, $\sum_{n=1}^{\infty} \frac{n}{1 + n^2}$ diverges (see Example 1). Let

$$f(x) = \frac{x}{(1 + x^2)^p},\ p \neq 1$$

$$f'(x) = \frac{1 - (2p - 1)x^2}{(1 + x^2)^{p+1}}.$$

For a fixed $p > 0$, $p \neq 1$, $f'(x)$ is eventually negative. f is positive, continuous, and eventually decreasing.

$$\int_1^{\infty} \frac{x}{(1 + x^2)^p}\, dx = \left[\frac{1}{(x^2 + 1)^{p-1}(2 - 2p)}\right]_1^{\infty}$$

For $p > 1$, this integral converges. For $0 < p < 1$, it diverges.

61. $\sum_{n=1}^{\infty} \frac{1}{p^n} = \sum_{n=1}^{\infty} \left(\frac{1}{p}\right)^n$, Geometric series.

Converges for $\left|\frac{1}{p}\right| < 1 \Rightarrow p > 1$