AP CALCULUS BC	LECTURE NOTES	MR. RECORD
Section Number: **9.4**	**Topics: Comparison of Series** - Direct Comparison Test - Limit Comparison Test	**Day:** 1 of 1

Direct Comparison Test

For the convergence tests that we have discussed so far, the terms of the series seem to be pretty simple and the series must have a special "look" to it in order for the convergence tests to be applied.
Any slight deviation from that "look" or characteristic, can make the convergence tests nonapplicable.

1. $\sum_{n=0}^{\infty}\frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty}\frac{n}{2^n}$ is not.
2. $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a p-series, but $\sum_{n=0}^{\infty}\frac{1}{n^3+1}$ is not.
3. $a_n=\frac{n}{\left(n^2+3\right)^2}$ is easily integrated, but $b_n=\frac{n^3}{\left(n^2+3\right)^2}$ is not.

Our goal now becomes to investigate a way to **compare** a series with complicated terms with a simpler series whose convergence or divergence is known.

> **THEOREM 9.11: DIRECT COMPARISON TEST**
> Let $0<a_n\le b_n$ for all n.
> 1. If $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.
> 2. If $\sum_{n=1}^{\infty}a_n$ diverges, then $\sum_{n=1}^{\infty}b_n$ diverges.

Example 1: Using the Direct Comparison Test.
Dertermine the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty}\frac{1}{2+3^n}$

This series resembles

$\sum_{n=1}^{\infty}\frac{1}{3^n}$ (a convergent geometric series)

Term by term comparison

$n=$	1	2	3	
$\frac{1}{2+3^n}$	$\frac{1}{5}$	$\frac{1}{11}$	$\frac{1}{29}$	...
$\frac{1}{3^n}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	...

$\Rightarrow a_n=\frac{1}{2+3^n}<\frac{1}{3^n}=b_n \quad n\ge 1$

So by the Direct Comparison Test, the series converges.

b. $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$

This series resembles

$\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ (a divergent p-series)

Term by term comparison

n	1	2	3	4
$\frac{1}{2+\sqrt{n}}$	$\frac{1}{3}$			$\frac{1}{4}$
$\frac{1}{\sqrt{n}}$	1			$\frac{1}{2}$

$\frac{1}{2+\sqrt{n}}\le\frac{1}{\sqrt{n}}$ for $n\ge 1$

DCT tells nothing

Using $\frac{1}{n}$ as your comparison series

$\frac{1}{n}\le\frac{1}{2+\sqrt{n}} \quad n\ge 4$

tells us the series diverges

Remember that both parts of the Direct Comparison Test require that $0<a_n\le b_n$.
Informally, the test says the following about the two series with nonnegative terms

1. **If the "larger" series converges, the "smaller" series must also converge.**
2. **If the "smaller" series diverges, then the "larger" series must also diverge.**

Limit Comparison Test

Often a series closely resembles a p-series or a geometric series, yet one cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test.
If this happens, it may be possible to apply a second type of test called the **Limit Comparison Test.**

> **THEOREM 9.12: LIMIT COMPARISON TEST**
> Suppose $a_n > 0,\ b_n > 0$, and
> $$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L \quad \text{where } L \text{ is } \textit{finite and positive.}$$
> Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 2: Using the Limit Comparison Test.
Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty}\frac{1}{an+b},\quad a>0,\ b>0$$

Let's compare with $\sum_{n=1}^{\infty}\frac{1}{n}$ (divergent harmonic series)

$$\lim_{n\to\infty}\frac{\frac{1}{an+b}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{an+b} = \frac{1}{a}$$

$\frac{1}{a} > 0$, Thus $\boxed{\text{the series diverges}}$

Helpful examples in choosing an appropriate *p*-series when using the Limit Comparison Test

General Series	Comparison Series	Conclusion
$\sum_{n=1}^{\infty}\frac{1}{3n^2-4n+5}$	$\sum_{n=1}^{\infty}\frac{1}{n^2}$	Both series converge.
$\sum_{n=1}^{\infty}\frac{1}{\sqrt{3n-2}}$	$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$	Both series diverge.
$\sum_{n=1}^{\infty}\frac{n^2-10}{4n^5+n^3}$	$\sum_{n=1}^{\infty}\frac{n^2}{n^5}=\sum_{n=1}^{\infty}\frac{1}{n^3}$	Both series converge.

Example 3: Using the Limit Comparison Test.
Determine the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

Compare with $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2}$ or $\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}}$ (convergent p-series)

$$= \lim_{n\to\infty}\frac{\sqrt{n}}{n^2+1}\cdot\frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty}\frac{n^2}{n^2+1}$$

$$= 1$$

b. $\sum_{n=1}^{\infty}\frac{n2^n}{4n^3+1}$

Compare with $\underbrace{\sum_{n=1}^{\infty}\frac{2^n}{n^2}}$ divergent series

$$= \lim_{n\to\infty}\frac{n\cdot 2^n}{4n^3+1}\cdot\frac{n^2}{2^n}$$

$$= \lim_{n\to\infty}\frac{n^3}{4n^3+1}$$

$$= \frac{1}{4}$$

$\boxed{\text{The series diverges}}$