AP CALCULUS BC	LECTURE NOTES	MR. RECORD
Section Number: **9.5**	**Topics: Alternating Series** - Alternating Series - Alternating Series Remainder	**Day: 1 of 2**

Alternating Series

Up to now, most series we have dealt with had positive terms. In this section and beyond, we'll study series that contain both positive and negative terms. The simplest of these series is called an **alternating series** whose terms alternate signs.

THEOREM 9.14: ALTERNATING SERIES TEST

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge if the following two conditions are met

1. $\lim\limits_{x\to\infty} a_n = 0$ **2.** If $a_{n+1} \le a_n$ for all n

Example 1: Using the Alternating Series Test.

Determine the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

1. $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

2. $\frac{1}{n+1} \le \frac{1}{n}$ for all n

$\boxed{\text{the series converges}}$

b. $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$

$n=$	1	2	3	4	5	6
$a_n=$	1	-1	$3/4$	$4/-8$	$5/16$	$-6/32$
S_n	1	0	$7/4$	$5/4$	$25/16$	$28/16$

$= \sum_{i=1}^{\infty}(-1)^{n+1}\frac{n}{2^{n-1}}$

1. $\lim \frac{n}{2^{n-1}} = 0$

2. $\frac{n+1}{2^n} \le \frac{n}{2^{n-1}}$ for $n \ge 1$

$\boxed{\text{converges}}$

Alternating Series Remainder

THEOREM 9.15: ALTERNATING SERIES REMAINDER

If a convergent alternating series satisfies the condition Let $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term That is,

$$|S - S_N| = |R_N| \le a_{N+1}$$

Calculator

Example 2: Approximating the Sum of an Alternating Series.

Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

1. $\lim\limits_{n\to\infty} \frac{1}{n!} = 0$

2. $\frac{1}{(n+1)!} \le \frac{1}{n!}$ for $n \ge 1$

$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720}$

$= \frac{91}{144} \approx 0.63194$

$a_7 = \frac{1}{5040} \approx 0.0002$

$|S - S_6| = |R_6| \le a_7$

$|S - 0.63194| \le 0.0002 \rightarrow \boxed{0.63174 \le S \le 0.63214}$

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Section Number: **9.5**	**Topics: Alternating Series** - Absolute and Conditional Convergence - Rearrangement of Series	**Day:** 2 of 2

Absolute and Conditional Convergence

There are times that we may encounter a series that has both positive and negative terms and yet it does not alternate.

For example, $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdot$

has both positive and negative terms but it does not alternate. An efficient way to investigate the convergence/divergence of this series is to consider this series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$$

By direct comparison, we see $|\sin n| \le 1$ for all n, so

$\left| \frac{\sin n}{n^2} \right| \le \frac{1}{n^2}, \quad n \ge 1$ This means that $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ converges.

The theorem that follows tells us that the original series also converges.

> **THEOREM 9.16: ABSOLUTE CONVERGENCE**
> If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Note: The converse of the above theorem is not true. For instance, the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called **conditional.**

> **DEFINITIONS OF ABSOLUTE AND CONDITIONAL CONVERGENCE**
> 1. $\sum a_n$ **is absolutely convergent if** $\sum |a_n|$ converges.
> 2. $\sum a_n$ **is conditionally convergent if** $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 3: Absolute and Conditional Convergence.

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} = \frac{0!}{2^0} - \frac{1!}{2^1} + \frac{2!}{2^2} - \frac{3!}{2^3} + \cdots$

$\lim_{n \to \infty} \frac{n!}{2^n} = \infty$ by n-th term test

$\boxed{\text{diverges}}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots$

1. $\lim_{n \to \infty} \frac{1}{n^{1/2}} = 0$
2. $\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$ for $n \ge 1$

The alternating series converges

but

$\frac{1}{n^{1/2}}$ is a divergent p-series

$\boxed{\text{conditionally convergent}}$

Rearrangement of Series

A finite sum such as (1 + 3 – 2 + 5 – 4) can be rearranged without changing the value of the sum. This, however, is not necessarily true of an infinite series—it depends on whether the series is absolutely convergent (where *every* rearrangement has the same sum) or conditionally convergent.

Example 3: Rearrangement of a Series.

The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges to $\ln 2$

That is to say $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2$

Rearrange the series to produce a different sum.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \frac{1}{1}-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\cdots$$

$$= \underbrace{\left(\frac{1}{1}-\frac{1}{2}\right)}-\frac{1}{4}+\underbrace{\left(\frac{1}{3}-\frac{1}{6}\right)}-\frac{1}{8}+\underbrace{\left(\frac{1}{5}-\frac{1}{10}\right)}-\frac{1}{12}+\underbrace{\left(\frac{1}{7}-\frac{1}{14}\right)}-\cdots$$

$$= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\cdots$$

$$= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right)$$

$$= \frac{1}{2}(\ln 2).$$