AP CALCULUS BC	LECTURE NOTES	MR. RECORD
Section Number: **9.6**	**Topics: The Ratio and Root Tests** The Ratio Test	**Day:** 1 of 2

The Ratio Test

Here is another test for absolute convergence – the **Ratio Test.**

THEOREM 9.17: RATIO TEST

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$.

3. The ~~Root~~ Ratio Test is inconclusive if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$.

Example 1: Using the Ratio Test.

Determine the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty}\frac{2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty}\frac{\underbrace{2^{n+1}}_{2^n\cdot 2}}{\underbrace{(n+1)!}_{(n+1)n!}}\cdot\frac{n!}{2^n} = \lim_{n\to\infty}\frac{2}{n+1} = 0$$

$\boxed{\text{converges absolutely}}$

b. $\sum_{n=1}^{\infty}\frac{n^2 2^{n+1}}{3^n}$

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2 2^{n+2}}{3^{n+1}}}{\frac{n^2\cdot 2^{n+1}}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^2\cdot \overbrace{2^{n+2}}^{2^n\cdot 2^2}}{3^{n+1}}\cdot\frac{3^n}{n^2\cdot \underbrace{2^{n+1}}_{2^n\cdot 2}} = \lim_{n\to\infty}\frac{(n+1)^2\cdot 2\cdot}{n^2\cdot 3} = \frac{2}{3}$$

$\boxed{\text{converges absolutely}}$

c. $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n(n+1)!} =$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$= e$ surprise!

$e>1$ $\boxed{\text{diverges}}$

d. $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{\sqrt{n+1}}{n+2}}{(-1)^n\frac{\sqrt{n}}{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{\sqrt{n+1}}{n+2}\cdot\frac{n+1}{\sqrt{n}}$$

$$= \lim_{n\to\infty}\sqrt{\frac{n+1}{n}}\cdot\frac{n+1}{n+2}$$

$= 1$

$\boxed{\text{Ratio Test is inconclusive}}$ (Try Alt. Ser. Test)

The Ratio Test

Here is another test for convergence and divergence – the **Root Test.** You will discover that it works especially well for series involving nth powers.

THEOREM 9.18: ROOT TEST

Let $\sum a_n$ be a series.

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Note: The Root Test is always inconclusive for any p-series.

Example 2: Using the Root Test.

Determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0$$

$\boxed{\text{converges}}$

Strategies for Testing Series

You have now studied 10 tests for determining the convergence or divergence of an infinite series. The following guidelines can help you get started.

GUIDELINES FOR TESTING A SERIES FOR CONVERGENCE OR DIVERGENCE

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types – geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Ratio Test or the Root Test be applied?
4. Can the series be compared favorably to one of the special types?

See additional handouts for a breakdown of each test.

Example 3: Applying Strategies for Testing Series.

Determine the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

by - nth term test

$\lim_{n\to\infty} \frac{n+1}{3n+1} = \frac{1}{3}$

$\boxed{\text{diverges}}$

b. $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$

geo-series $r = \frac{\pi}{6} < 1$

$\boxed{\text{converges}}$

c. $\sum_{n=1}^{\infty} ne^{-n^2}$

ne^{-n^2} pos, cont.

$f' = ne^{-n^2}(-2n) + 1e^{-n^2}$

$= e^{-n^2}(-2n^2+1) < 0$ for $n > 1$

f is dec.

So, by the integral test

$\int_0^{\infty} ne^{-n^2}\,dx$

$-\frac{1}{2}e^{-n^2}\Big|_0^{\infty}$

$-\frac{1}{2e^{n^2}}\Big|_0^{\infty} = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$

$\boxed{\text{Converges}}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

compare to $\frac{1}{3n}$ a divergent p-series

$\frac{1}{3n+1} < \frac{1}{3n}$

div

this is inconclusive

By Lim Comp Test

$\lim_{n\to\infty} \frac{\frac{1}{3n+1}}{\frac{1}{n}}$; $\lim_{n\to\infty} \frac{1}{3n+1} \cdot \frac{n}{1} = \frac{1}{3}$ finite, pos

$\boxed{\text{diverges}}$

e. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$

By Alt Ser Test,

1. $\lim_{n\to\infty} \frac{3}{4n+1} = 0$ ✓

$\frac{3}{4n+5} \overset{?}{\leq} \frac{3}{4n+1} \rightarrow 4n+1 \overset{?}{\leq} 4n+5$ yes ✓

$\boxed{\text{converges}}$

f. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

By Ratio Test

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!}$

$= \lim_{n\to\infty} \frac{n+1}{10} = \infty$

$\boxed{\text{diverges}}$

g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n+1}{2n+1}\right)^{n/n} = \frac{1}{2}$

$\boxed{\text{converges}}$