13. $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/2^{n+1}}{1/2^n}\right| = \lim_{n\to\infty}\frac{2^n}{2^{n+1}} = \frac{1}{2} < 1$$

Therefore, the series converges by the Ratio Test.

15. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

17. $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(6/5)^{n+1}}{n(6/5)^n}$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{6}{5}\right) = \frac{6}{5} > 1$$

Therefore, the series diverges by the Ratio Test.

19. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)/4^{n+1}}{n/4^n} = \lim_{n\to\infty}\frac{n+1}{4n} = 1/4 < 1$$

Therefore, the series converges by the Ratio Test.

21. $\sum_{n=1}^{\infty} \frac{4^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4^{n+1}/(n+1)^2}{4^n/n^2}$$

$$= \lim_{n\to\infty}\frac{n^2}{(n+1)^2}(4) = 4 > 1$$

Therefore, the series diverges by the Ratio Test.

23. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

25. $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)3^{n+1}}\cdot\frac{n3^n}{n!}\right| = \lim_{n\to\infty}\frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

27. $\sum_{n=0}^{\infty} \frac{e^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}/(n+1)!}{e^n/n!}$$

$$= \lim_{n\to\infty} e\left(\frac{n!}{(n+1)!}\right) = \lim_{n\to\infty}\frac{e}{n+1} = 0$$

Therefore, the series converges by the Ratio Test.

29. $\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{6^{n+1}/(n+2)^{n+1}}{6^n/(n+1)^n} = \lim_{n\to\infty}\frac{6}{n+2}\left(\frac{n+1}{n+2}\right)^n = 0\left(\frac{1}{e}\right) = 0.$$

To find $\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$: Let $y = \left(\frac{n+1}{n+2}\right)^n$

$$\ln y = n\ln\left(\frac{n+1}{n+2}\right) = \frac{\ln(n+1) - \ln(n+2)}{1/n}$$

$$\lim_{n\to\infty}[\ln y] = \lim_{n\to\infty}\left[\frac{1/(n+1) - 1/(n+2)}{-1/n^2}\right] = \lim_{n\to\infty}\left[\frac{-n^2[(n+2)-(n+1)]}{(n+1)(n+2)}\right] = -1$$

by L'Hôpital's Rule. So, $y \to \frac{1}{e}$.

Therefore, the series converges by the Ratio Test.

31. $\sum_{n=0}^{\infty} \frac{5^n}{2^n + 1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5^{n+1}/(2^{n+1}+1)}{5^n/(2^n+1)} = \lim_{n\to\infty} \frac{5(2^n+1)}{(2^{n+1}+1)} = \lim_{n\to\infty} \frac{5(1+1/2^n)}{2+1/2^n} = \frac{5}{2} > 1$$

Therefore, the series diverges by the Ratio Test.

33. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$.

35. $\sum_{n=1}^{\infty} \frac{1}{5^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left[\frac{1}{5^n}\right]^{1/n} = \frac{1}{5} < 1$$

Therefore, by the Root Test, the series converges.

37. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

39. $\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{n-1}\right)^n} = \lim_{n\to\infty} \left(\frac{2n+1}{n-1}\right) = 2$$

Therefore, by the Root Test, the series diverges.

41. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \frac{1}{|\ln n|} = 0$$

Therefore, by the Root Test, the series converges.

43. $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n} + 1\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(2\sqrt[n]{n} + 1\right)^n} = \lim_{n\to\infty} \left(2\sqrt[n]{n} + 1\right)$$

To find $\lim_{n\to\infty} \sqrt[n]{n}$, let $y = \lim_{n\to\infty} \sqrt[n]{n}$. Then

$$\ln y = \lim_{n\to\infty} \left(\ln \sqrt[n]{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

So, $\ln y = 0$, so $y = e^0 = 1$ and

$$\lim_{n\to\infty} \left(2\sqrt[n]{n} + 1\right) = 2(1) + 1 = 3.$$

Therefore, by the Root Test, the series diverges.

45. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{3^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{3} = \frac{1}{3}$$

Therefore, the series converges by the Root Test.

Note: You can use L'Hôpital's Rule to show

$\lim_{n\to\infty} n^{1/n} = 1$:

Let $y = n^{1/n}$, $\ln y = \frac{1}{n} \ln n = \frac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \Rightarrow y \to 1$$

47. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = 0 - 0 = 0 < 1$$

Therefore, by the Root Test, the series converges.

49. $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty}\frac{n^{1/n}}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.