Polynomial Approximations of Elementary Functions

Goal: Show how polynomial functions can be used as approximations for other elementary functions.

We want to find a polynomial \( P \) that approximates another polynomial \( f \).

Step 1: We begin by choosing a number \( c \) in the domain of \( f \) at which \( f \) and \( P \) have the same value.

Therefore, \( P(c) = f(c) \).

The approximating polynomial, \( P \), is said to be **expanded about \( c \) or centered at \( c \).**

Step 2: Our task is now to “build” a polynomial, \( P \), whose graph resembles that of \( f \) near the point above.

We do this by imposing additional requirements that the slope of the polynomial be the same as that of the graph of \( f \) near \( (c, f(c)) \).

Therefore, \( P'(c) = f'(c) \).

**Example 1: First Degree Polynomial Approximation of \( f(x) = e^x \).**

For the function \( f(x) = e^x \), find a first degree polynomial function \( P_1(x) = a_0 + a_1x \) whose value and slope agree with the value and slope of \( f \) at \( x = 0 \).

\[
\begin{align*}
  f(0) &= e^0 = 1 \\
  P_1(0) &= 1 \\
  P_1'(0) &= e^0 = 1 \\
  P_1(x) &= 1 + x \\
  a_0 &= 1 \\
  a_1 &= 1
\end{align*}
\]

**Activity: Use your TI-Nspire.**

1. Sketch of graph of \( f \) and \( P \) on the same coordinate plane using a relatively small window centered around the point \((0, 1)\).

Notice how the two graphs move farther away from each other as we move farther from \((0, 1)\).

To improve the approximation, we could impose yet another requirement – that the values of the second derivatives of \( f \) and \( P \) agree when \( x = 0 \). The polynomial, \( P_2 \), of least degree that satisfies all three requirements:

\[
P_2(0) = f(0), \quad P'_2(0) = f'(0), \quad \text{and} \quad P''_2(0) = f''(0) \text{ can be shown to be}
\]

\[
P_2(x) = 1 + x + \frac{1}{2} x^2
\]

2. Sketch of graph of \( P_2(x) \) on the same coordinate plane containing the sketch from 1 above.

Continuing this pattern of requiring that the values of \( P_n(x) \) and its first \( n \) derivatives match those of \( f(x) = e^x \) at \( x = 0 \), we would obtain the following

\[
P_n(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots + \frac{1}{n!} x^n
\]

**n-th degree approximation**
Taylor and Maclaurin Polynomials

The polynomial approximations of \( f(x) = e^x \) given in our previous examples are each expanded about \( x = 0 \). For expansions about an arbitrary value of \( c \), it is convenient to write the polynomial in the form

\[
P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n
\]

In this form, repeated differentiation produces

\[
\begin{align*}
P_n'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + n a_n(x-c)^{n-1} \\
P_n''(x) &= 2a_2 + 2(3a_3)(x-c) + \cdots + n(n-1)a_n(x-c)^{n-2} \\
P_n'''(x) &= 2(3a_3) + \cdots + n(n-1)(n-2)a_n(x-c)^{n-3}
\end{align*}
\]

\[
P_n^{(n)}(x) = n(n-1)(n-2) \ldots (2)(1)a_n
\]

Letting \( x = c \), we then obtain

\[
P_n^{(n)}(c) = n! a_n
\]

\[
P_n(c) = a_0, \quad P_n'(c) = a_1, \quad P_n''(c) = 2a_2, \ldots, \quad \frac{f^{(n)}(c)}{n!} = a_n
\]

With these coefficients, we can obtain the following definition of Taylor polynomials, named after the English mathematician Brook Taylor, and Maclaurin polynomials, named after English mathematician, Colin Maclaurin (1698-1746).

### DEFINITION OF \( n \)TH TAYLOR POLYNOMIAL AND \( n \)TH MACLAURIN POLYNOMIAL

If \( f \) has \( n \) derivatives at \( c \), then the polynomial

\[
P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n
\]

is called the \( n \)th Taylor polynomial for \( f \) centered at \( c \).

If \( c = 0 \), then

\[
P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n
\]

is also called the \( n \)th Maclaurin polynomial for \( f \).

### Example 2: A Maclaurin Polynomial for \( f(x) = e^x \)

Find the \( n \)th Maclaurin polynomial for \( f(x) = e^x \).

\[
P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}
\]
Example 3: Finding Taylor Polynomials for $f(x) = \ln x$

Find the Taylor polynomials $P_0, P_1, P_2, P_3,$ and $P_4$ for $f(x) = \ln x$ centered at $c = 1$

\[ f(x) = \ln x \quad \quad f(1) = \ln 1 = 0 \]
\[ f'(x) = \frac{1}{x} \quad \quad f'(1) = 1 \]
\[ f''(x) = -\frac{1}{x^2} \quad \quad f''(1) = -1 \]
\[ f'''(x) = \frac{2}{x^3} \quad \quad f'''(1) = 2 \]
\[ f^{(4)}(x) = -\frac{6}{x^4} \quad \quad f^{(4)}(1) = -6 \]

\[ P_0(x) = f(1) = 0 \]
\[ P_1(x) = f(1) + f'(1)(x-1) = x - 1 \]
\[ P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} = x - 1 - \frac{1}{2}(x-1)^2 \]
\[ P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \]
\[ = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \]
\[ P_4(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \]
### Example 4: Finding Maclaurin Polynomials for $f(x) = \cos x$

Find the Maclaurin polynomials $P_0, P_2, P_4,$ and $P_6$ for $f(x) = \cos x$.

Use $P_6(x)$ to approximate the value of $\cos(0.1)$

\[ f(x) = \cos x \quad f(0) = 1 \quad P_6(x) = 1 \]
\[ f'(x) = -\sin x \quad f'(0) = 0 \quad P_2(x) = 1 - \frac{1}{2}x \]
\[ f''(x) = -\cos x \quad f''(0) = -1 \quad P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \]
\[ f'''(x) = \sin x \quad f'''(0) = 0 \quad P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \]
\[ f^{(4)}(x) = \cos x \quad f^{(4)}(0) = 1 \]
\[ f^{(5)}(x) = -\sin x \quad f^{(5)}(0) = 0 \]
\[ f^{(6)}(x) = -\cos x \quad f^{(6)}(0) = -1 \]

\[ P_6(0.1) \approx 0.995004165 \]

**Note:** $\cos(0.1) \approx 0.995004165$

What powers of $x$ are present for the Maclaurin series for $\cos x$?

### Example 5: Finding a Taylor Polynomial for $f(x) = \sin x$

Find the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \frac{\pi}{6}$.

\[ f(x) = \sin x \quad f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2} \]
\[ f'(x) = \cos x \quad f'\left(\frac{\pi}{6}\right) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \]
\[ f''(x) = -\sin x \quad f''\left(\frac{\pi}{6}\right) = -\sin \frac{\pi}{6} = -\frac{1}{2} \]
\[ f'''(x) = -\cos x \quad f'''\left(\frac{\pi}{6}\right) = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2} \]

\[ P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) - \frac{1}{2 \cdot 2!} \left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2 \cdot 3!} \left(x - \frac{\pi}{6}\right)^3 \]

Compare $f(x) = \sin x$ and $P_3(x)$ on your TI-Nspire.
Example 6: Approximating Using Maclaurin Polynomials

a. Use a fourth Maclaurin polynomial to approximate the value of \( \ln(1.1) \).

\[
\begin{align*}
f(x) &= \ln(1+x) \\
f'(x) &= \frac{1}{1+x} \\
f''(x) &= -\frac{1}{(1+x)^2} \\
f'''(x) &= \frac{2}{(1+x)^3} \\
f^{(4)}(x) &= -\frac{6}{(1+x)^4}
\end{align*}
\]

\[
P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4
\]

\[
= 0 + 1 - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4
\]

\[
= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4
\]

use 0.1

\[P_4(0.1) \approx 0.095308\]

\[\ln(1.01) \approx 0.09531\]

b. Complete the table rounding your results to six decimal places.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0.1</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>ln(1 + x)</td>
<td>0</td>
<td>0.095310</td>
<td>0.405465</td>
<td>0.559616</td>
<td>0.693147</td>
</tr>
<tr>
<td>(P_4(x))</td>
<td>0</td>
<td>0.095308</td>
<td>0.401042</td>
<td>0.530273</td>
<td>0.583333</td>
</tr>
</tbody>
</table>

**KEY POINTS**

1. The approximation is usually better at \( x \)-values close to \( c \) than at \( x \)-values far from \( c \).

2. The approximation is usually better for higher-degree Taylor (or Maclaurin) polynomials than for those of lower degree.
The Remainder of a Taylor Polynomial

\[ f(x) = P_n(x) + R_n(x) \]

The remainder will serve as the error in our approximation, therefore,

\[ \text{Error} = |R_n(x)| = |f(x) - P_n(x)| \]

The theorem below gives a general procedure for estimating the remainder associated with a Taylor polynomial. The theorem, called Taylor's Theorem, defines a remainder that is often called the Lagrange form of the remainder.

**THEOREM 9.19 TAYLOR'S THEOREM**

If a function \( f \) is differentiable through order \( n + 1 \) in an interval \( I \) containing \( c \), then, for each \( x \) in \( I \), there exists \( z \) between \( x \) and \( c \) such that

\[ f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x) \]

where

\[ R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \]

Example 7: Determining the Accuracy of the Approximation

The third Maclaurin polynomial for \( \sin x \) is given by \( P_3(x) = x - \frac{x^3}{3!} \) \( x = 0.1 \) \( \text{Maclaurin} \).

Use Taylor's Theorem to approximate \( \sin(0.1) \) by \( P_3(0.1) \) and determine the accuracy of the approximation.

\[ \sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!}x^4 \]

\( 0 < z < 0.1 \)

\[ \sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.1 - 0.0001666667 \approx 0.0998333333 \]

\[ f^{(4)}(z) = \sin z \]

So,

\[ 0 < |R_3(0.1)| = \frac{\sin z}{4!}(0.1)^4 < \frac{0.0001}{4!} \approx 0.0000041667 \]

\[ 0.099833333 < \sin(0.1) = 0.0998333333 + R_3(x) < 0.0998333333 + 0.0000041667 \]

\[ 0.099833 < \sin(0.1) < 0.0998375 \]
Example 8: Approximating a Value to a Desired Accuracy

Determine the degree of the Taylor polynomial $P_n(x)$ expanded about $c = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.001.

The $(n+1)$st derivative of $f(x) = \ln x$ is

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}$$

$$|R_n(1.2)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (1.2 - 1)^{n+1} \right| = \frac{n!}{z^{n+1}} \frac{1}{(n+1)!} (0.2)^{n+1}$$

$$= \frac{(0.2)^{n+1}}{z^{n+1} (n+1)}$$

where $1 < z < 1.2$

In this interval

$$\frac{(0.2)^{n+1}}{z^{n+1} (n+1)} < \frac{(0.2)^{n+1}}{n+1}$$

$$> \frac{1}{n}$$

So we seek an $n$ such that

$$\frac{(0.2)^{n+1}}{n+1} < 0.001$$

$$\frac{n+1}{(0.2)^{n+1}} > 1000$$

$$(n+1)(5^{n+1}) > 1000$$

By Trial and Error

$n = 2 \quad 3 \quad (5^3) = 375$

$n = 3 \quad 4 \quad (5^4) = 40625 = 25000$

You would need the 3rd Taylor Polynomial