Avon High School **Name** KEY

AP Calculus BC

UNIT 6 EXAM REVIEW

Sections 9.1-9.6: Sequences and Infinite Series

For Problems 1-5, be sure to justify your conclusion by thoroughly demonstrating the test(s) you use.

1. Does the sequence $a_n = \left\{\dfrac{3n}{2n-1}\right\}$ converge or diverge?

$$\lim_{n\to\infty} \frac{3n}{2n-1} = \frac{3}{2}$$

Since the limit of our sequence exists, our sequence $\boxed{\text{converges}}$.

2. Does the series $\sum_{n=1}^{\infty} \dfrac{3n}{2n-1}$ converge or diverge?

Using the n-th term test,

$$\lim_{n\to\infty} \frac{3n}{2n-1} = \frac{3}{2} \neq 0.$$

Therefore, this series $\boxed{\text{diverges}}$.

3. Use the n-th term test on the series $\sum_{n=1}^{\infty} \dfrac{6n}{4n^2+1}$ and state its conclusion.

$$\lim_{n\to\infty} \frac{6n}{4n^2+1} = 0$$

The n-th term test is $\boxed{\text{inconclusive}}$.

4. Can you find the sum of the series $\sum_{n=1}^{\infty} \dfrac{2^n}{5^{n+1}}$? If so, find that sum.

Rewrite as $\sum_{n=1}^{\infty} \dfrac{2^n}{5^n \cdot 5} = \sum_{n=1}^{\infty} \dfrac{1}{5}\left(\dfrac{2}{5}\right)^n$. Hey! It's a geometric series. . . . . . (sort of)

Change to $n=0$ $\Rightarrow \sum_{n=0}^{\infty} \dfrac{1}{5}\left(\dfrac{2}{5}\right)^{n+1} = \sum_{n=0}^{\infty} \dfrac{2}{25}\left(\dfrac{2}{5}\right)^n$

$$S = \frac{\frac{2}{25}}{1-\frac{2}{5}} = \frac{2}{25}\cdot\frac{5}{3}$$

$$= \boxed{\frac{2}{15}}$$

5. Given the series $\sum_{n=1}^{\infty}\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$, find its sum.

$$= \left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1}-\frac{1}{n+1}\right) + \left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow S = 1 + \frac{1}{2} + \lim_{n\to\infty}\left(-\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \boxed{\frac{3}{2}}$$

6. Match the series with its behavior by circling the appropriate word.

a. $\sum_{n=1}^{\infty} \frac{1}{n^e}$ converge p-series where $p = e > 1$ diverge

b. $\sum_{n=1}^{\infty} \frac{1}{n^{\ln 2}}$ converge diverge p-series where $p = \ln 2 < 1$

(Note $\ln e = 1$)
$\therefore \ln 2 < 1$

7. Find the value of r (if it exists) in the series $3 - \frac{9}{2} + \frac{27}{4} - \cdots$

a. $-\frac{3}{2}$ **b.** $-\frac{2}{3}$ **c.** $\frac{2}{3}$ **d.** $\frac{3}{2}$ **e.** no r exists

a_2/a_1

$$\frac{-\frac{9}{2}}{3} = -\frac{9}{2}\cdot\frac{1}{3} = -\frac{3}{2};$$

Double check: a_3/a_2

$$\frac{\frac{27}{4}}{-\frac{9}{2}} = \frac{27}{4}\cdot -\frac{2}{9} = -\frac{3}{2}$$

8. How many terms of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$ are needed to approximate the sum of the series with an error that is less than 0.001?

a. 3 **b.** 4 **c.** 5 **d.** 6 **e.** 7

$|S - S_n| \le |a_{n+1}|$

Term #:	3	4	5	6
a :	$\frac{1}{81}$	$\frac{1}{256}$	$\frac{1}{625}$	$\frac{1}{1296}$

This would represent the first "neglected term."
This is also the error.

$$\begin{array}{r} {}^{3}36 \\ \times\ 36 \\ \hline 216 \\ 108 \end{array}$$

9. If the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ converges, which of the following series also converges by the Direct Comparison Test?

a. $\sum_{n=1}^{\infty} \frac{1}{n}$ **b.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **c.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{1.01}}$ **d.** $\sum_{n=1}^{\infty} \frac{1}{n^{0.09}}$ **e.** $\sum_{n=1}^{\infty} \frac{1}{n^{0.01}}$

I'm not really sure you'd need to use the D.C.T here.
The series for a, b, d and e are obviously divergent.
Choice c satisfied the criteria for convergence using the Alternating Series Test

1. $\lim_{n\to\infty} \frac{1}{n^{1.01}} = 0$

2. $\frac{1}{n^{1.01}}$ is decreasing for $n > 1$

$$\frac{1}{(n+1)^{1.01}} \overset{?}{<} \frac{1}{n^{1.01}}$$

$$n^{1.01} < (n+1)^{1.01} \checkmark$$

$$+\left(\frac{1}{n-2}-\frac{1}{n}\right)+$$

10. Find the sum of the series $\cancel{\left(1-\frac{1}{3}\right)}+\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)$

a. $\frac{2}{3}$ **b.** $\frac{4}{5}$ **c.** $\frac{6}{7}$ **d.** 1 **e.** does not exist

$$S = 1 - \lim_{n\to\infty}\frac{1}{n+2} = 1$$

11. Which of the following series is <u>absolutely</u> convergent?

~~a.~~ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{0.99}}$ Similar to choice (c). See below.

b. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. Use A.S.T. on $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

1. $\lim_{n\to\infty}\frac{1}{n^2}=0$

2. $\frac{1}{(n+1)^2} \overset{?}{<} \frac{1}{n^2}$

$n^2 < (n+1)^2$ ✓

∴ This series is absolutely convergent.

~~c.~~ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ We studied this in class. It is conditionally convergent

~~d.~~ $\sum_{n=1}^{\infty}(-1)^{n+1}3^n$ Not a chance this is convergent.

~~e.~~ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln n}$ I think this is a trick. This series is undefined for $n=1$.
Now, if this choice had read $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln(n)}$, it would have been conditionally convergent.

12. Which of the following series is conditionally convergent?

~~a.~~ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ See Problem 11.

b. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ divergent p-series. Now use A.S.T.

1. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ ✓

2. $\frac{1}{\sqrt{n}}$ decreases

$\frac{1}{\sqrt{n+1}} \overset{?}{<} \frac{1}{\sqrt{n}}$ ✓

$\sqrt{n} < \sqrt{n+1}$ for $n \geq 1$

∴ converges by AST

~~c.~~ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$ Similar to choice (a)

~~d.~~ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$ Definitely absolutey convergent.

~~e.~~ $\sum_{n=1}^{\infty}(-1)^{n+1}e^n$ Diverges

Thus, conditional convergnce is shown.

13. $\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n =$

a. $\frac{2}{5}$ b. $\frac{2}{3}$ c. 1 d. $\frac{5}{3}$ e. $\frac{5}{2}$

This is a geometric series where $a=1$, $r=\frac{2}{5}$

$$S = \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

14. If $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, which of the following series also diverges?

a. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ No way, José.

b. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ We studied this in class. The alternating harmonic series converges

c. $\sum_{n=1}^{\infty}\frac{0.5}{n}$ This is really similar to $\sum_{n=1}^{\infty}\frac{1}{n}$. Multiplying a divergent series by a constant won't change the fact that it's divergent.

d. $\sum_{n=1}^{\infty}\frac{10^6}{n^2}$ p-series

e. $\sum_{n=1}^{\infty}\frac{1}{n^n}$ root test works well

convergent

15. Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}$. (DO NOT USE A CALCULATOR)

a. Determine the values of p for which the series converges. Justify your conclusion.

By the Alternating Series Test,

1. $\lim_{n\to\infty}\frac{1}{n^p}$ must be 0

This can only happen as long as p remains a positive value.

$\therefore \boxed{p>0}$

b. Find the maximum value of the error if k terms are used to approximate the sum of the series.

$$|S-S_n| \le \underbrace{|a_{n+1}|}_{\text{Error}}$$

$$\boxed{\frac{1}{(K+1)^p}}$$

* Replace n with K
Remember a_{K+1} is the first neglected term a.k.a. the error.

PROBEMS 16-18 REQUIRE THE USE OF A CLCULATOR.

16. a. Find the sum $\sum_{n=1}^{10} \frac{(-1)^{n+1}}{n^n}$.

This must be computed by calculator.
(Please don't try to write as a fraction.)

$\boxed{S_{10} \approx 0.783}$

b. What is the maximum value of the error if the sum $\sum_{n=1}^{10} \frac{(-1)^{n+1}}{n^n}$ is approximated by the answer in part (a)?

$|a_{n+1}| \Rightarrow |a_{11}| = \frac{1}{11^{11}}$ or $\approx 0.0000000000035049$ or 3.5049×10^{-12}

or $\frac{1}{285311670611}$

c. If the sum of the first twenty terms was used instead in part (a), would the error in part (b) be increased or decreased? Explain your answer.

Well, let's compute it. (Aferall we're allowed to use our TI-Nspire.)

$a_{21} \approx 1.35 \times 10^{-22}$

The error is $\boxed{\text{decreased}}$ because the value of a_n is decreasing.

17. a. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ correct to three decimal places. State the numer of terms needed to arrive at that answer.

This means ≤ 0.001

Error $= |a_{n+1}|$ where $|S - S_n| \leq |a_{n+1}|$

$n=$	1	2	$\cdots$	31	32
$a_n=$	1	$\frac{1}{4}$		$\frac{1}{961}$	$\underbrace{\frac{1}{1089}}_{\leq 0.001}$

$\therefore S_{31}$ will give a sum correct within 0.001.

$S_{31} = \sum_{n=1}^{31} \frac{1}{n^2} \approx \boxed{1.613}$; $\boxed{31 \text{ terms}}$

b. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ correct to three decimal places. State the numer of terms needed to arrive at that answer.

$n=$	1	2	$\cdots$	6	7
$a_n=$	1	$\frac{1}{2}$	$\cdots$	$\frac{1}{720}$	$\underbrace{\frac{1}{5040}}_{\leq 0.001}$

$\sum_{n=1}^{6} \frac{1}{n!} \approx \boxed{1.718}$ or exactly $\boxed{\frac{1237}{720}}$

$\boxed{6 \text{ terms}}$

18. **a.** Find the error when the alternating series $\sum_{n=0}^{\infty}(-0.25)^n$ is approximated by the first four terms of the series.

$$|a_{n+1}| = |a_5| = (-0.25)^4 = \left(-\frac{1}{4}\right)^4 = \boxed{\frac{1}{256} \approx 0.003906}$$

where $n=4$

b. Find the exact sum of the series and use it to find the actual error between the exact sum and the approximate sum using the first four terms of the series. Compare your answer to that in part (a).

$r = -0.25 \quad a = 1$ in this geometric series

$$S = \frac{1}{1-(-0.25)} = \frac{1}{1+\frac{1}{4}} = \frac{1}{\frac{5}{4}} = \frac{4}{5} = \boxed{0.8} \leftarrow \text{Exact Sum}$$

Will give the sum of the first four terms

$$\sum_{n=0}^{3}(-0.25)^n = \boxed{0.796875} \leftarrow \text{Partial sum}$$

$$|S-S_3| = |0.8-0.796875| = 0.003125 \text{ which is } \boxed{\text{less than}} \text{ } 0.003906$$