Chapter 5: Normal Probability Distributions

5.1 Introduction to Normal Distributions and the Standard Normal Distribution

What is a Normal Distribution and a Normal Curve?

A normal distribution is a probability distribution for a continuous random variable, x.

KEY FEATURES of NORMAL DISTRIBUTION
1. Mean, Median, and Mode are equal.
2. The curve is bell-shaped and symmetrical about the mean.
3. The total area under the curve = 1.
4. The curve approaches, but never touches the x-axis.
5. The graph curves downward within one standard deviation of the mean and curves upward outside one standard deviation of the mean. These points are called “inflection points.”

A normal curve can have any mean and any positive standard deviation.

\[ \mu = 3.5 \quad \sigma = 1.5 \]
\[ \mu = 3.5 \quad \sigma = 0.7 \]
\[ \mu = 1.5 \quad \sigma = 0.7 \]

Recall the mean is a measure of position: Curves A and B have the same mean.
Recall the standard deviation is a measure of spread: Curve A has the largest standard deviation while B and C have the same standard deviation.
Example 1:

Which curve has the greater mean?

Which curve has the greater standard deviation?

REVIEW CONCEPT

In Chapter 2, we used the EMPIRICAL RULE (see your formula sheet) to determine probabilities under normal curves.

Example 2: The birth weight of newborns is normally distributed with a mean of 3300 grams and a standard deviation of 600 grams.

a) Estimate the percentage of newborns with birth weights between 2100g and 4500g.

b) Estimate the percentage of newborns with birth weights less than 2700g.

In Chapter 2, we also discussed the “STANDARD SCORE” or “Z-SCORE.”
A z-score tells us “how many standard deviations” a raw score, x, is away from the mean, μ.

c) What is the Z score for a newborn weight of x = 3900g?

d) What is the z score for a newborn weight of x = 2300g?
A “Standard Normal Distribution” is a normal distribution with a mean of 0 and a standard deviation of 1. The total area under the curve is 1.

If we are dealing with values that have integer z-scores (-2, 0, 1, etc), we can still use the empirical rule to estimate probabilities.

If we are dealing with ANY z-score (-2, 0, 1.35, -0.78, etc.) we can use a “Standard Normal Distribution Table”

\[ z = \frac{\text{Value} - \mu}{\sigma} \]

### Standard Normal Distribution Table:

**Example:** Find the area left of \( z = -2.94 \)

**Example:** Find the area left of \( z = 0.37 \)
BIG IDEA IN THIS CHAPTER:
GOAL: Estimate probabilities of values of x that are part of a normal distribution.

PROCESS:
* Convert x-values (raw scores) to standard scores (z scores).
* Use the standard normal distribution table to look up areas. Areas under the normal curve correspond to probabilities!

HOW TO USE THE TABLE TO FIND PROBABILITIES:
FIRST: Sketch the curve and shade the appropriate areas
SECOND: Looking for the area on the...

LEFT SIDE:
Look up the value that corresponds to z in the table. This is the area to the left of z.

RIGHT SIDE:
Look up the value that corresponds to z in the table. This is the area to the left of z.

Subtract the area to left from 1.

1 – (area to the left) = area to the right

BETWEEN TWO VALUES
Find the area left of both scores. Subtract the smaller area from the larger area.

(larger area) – (smaller area) = (middle area)
Example 3: Find the area of the indicated region under the standard normal curve.

a) to the left of $z = 1.46$

b) to the left of $z = -2$

c) to the right of $z = -0.45$

d) to the right of $z = 1.9$

e) between $z = -1.43$ and $z = 2.03$

f) between $z = 0$ and $z = 1.79$
5.2 Normal Distributions: Finding Probabilities

Normal Curve vs. Standard Normal Curve

Normal curves:
Show mean and standard deviation values

Standard Normal curves:
Show z-scores (#standard deviations from the mean)

Can only estimate probabilities using empirical rule for SOME values of x.
Has a table of areas for EVERY z value (to 2 dec) to reference to find probabilities.

In other words:

\[ P(8.2 \leq x \leq 12.5) = \]

\[ P(-0.9 \leq z \leq 1.25) = \]

\[ z = \frac{x - \mu}{\sigma} \]

Example 1: A survey indicates that for each trip to the supermarket, a shopper spends 45 minutes with a standard deviation of 12 minutes in the store. The times are normally distributed and are represented by the variable x.

a) Find the probability that the customer spends less than 40 minutes in the supermarket.

\[ P(x < 40) = \]
A survey indicates that for each trip to the supermarket, a shopper spends 45 minutes with a standard deviation of 12 minutes in the store. The times are normally distributed and are represented by the variable \( x \).

b) Find the probability that the customer spends more than 55 minutes in the supermarket.

\[ P(x > 55) = \]

c) Find the probability that the customer spends between 33 and 60 minutes in the supermarket.

\[ P(33 < x < 60) = \]

Example 2: (pg 249 #7) In a survey of US men, the heights in the 20 – 29 age group were normally distributed with a mean of 69.4 inches and a standard deviation of 2.9 inches. Find the probability that a randomly selected participant has a height that is:

a) Less than 66 inches  

b) more than 72 inches  

c) between 66 and 72 inches
5.3 Normal Distributions: Finding Values

Sometimes we want to determine a RAW SCORE “X” that satisfies some condition about the percentage of data under the normal curve.

What’s the lowest test score can I receive and still be in the top 10% on a college entrance exam?

I want to do a study on the health habits of patients as pertains to weight. What weights are in the middle 60% of my patients?

What “warranty guarantee period“ should I advertise so I only have to refund the bottom 5% of all products purchased?

Finding a Z-Score Given An Area

AREA TO THE LEFT

Ex. 1
Find the z-score that corresponds to

a) 1.54% of the area to the left

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b) .0076 of the area to the left

c) 1.64% of the area to the left
AREA TO THE RIGHT

Ex. 2
Find the z-score that corresponds to

a) 20.33% to the right

b) 10% to the right

c) 0.4701 to the right

CENTERED AROUND THE MEAN

Ex 3: Find the z scores that correspond to the middle 98% of data.
Inverse Normal Distribution

Example 4: Magic Video Games, Inc., sells an expensive video games package. Because the package is so expensive, the company wants to advertise an impressive guarantee for the life expectancy of the computer system. The guarantee policy will refund the full purchase price if the computer fails during the guarantee period. The research department has done tests that show that the mean life for the computer is 30 months, with the standard deviation of 4 months. The computer life is normally distributed. How long can the guarantee period be if management does not want to refund the purchase price on more than 7% of the Magic Video Packages?

Example 5: (pg 259 #39) The weights of bags of baby carrots are normally distributed with a mean of 32 ounces and a standard deviation of 0.36 ounce. Bags in the upper 4.5% are too heavy and must be repackaged. What is the most a bag of baby carrots can weigh and not need to be repackaged?

How to find $x$, given a desired probability.

1. Determine what percentage of area under the normal curve you ACTUALLY want based on left/right/between and your given probability.
2. Use your table to estimate $z$.
3. Translate your $z$ answer into $x$ using the equation: $x = z\sigma + \mu$
5.4 Sampling Distributions and the Central Limit Theorem

*Random Number Table*

| 19267 | 95457 | 53497 |
| 71549 | 44843 | 26104 |
| 27386 | 50004 | 05358 |
| 76612 | 39789 | 13537 |
| 90379 | 51392 | 55887 |
| 94456 | 48396 | 73780 |
| 94730 | 95761 | 75023 |
| 93621 | 66330 | 33939 |
| 34449 | 63513 | 83834 |
| 73920 | 56297 | 72678 |

Mean: $\mu = 4.787$

Randomly choose a block of 5 numbers. Circle it.

Find the mean of your sample: $\bar{x} =$

A sampling distribution is the probability distribution of a sample statistic that is formed when sample of size $n$ are repeatedly taken from a population. In our case we would be looking at a “sampling distribution of sample means”.

The “mean of the means” (The average of our classroom results) would be written:

$$\mu_{\bar{x}}$$

The standard deviation of our of the sample means would be written:

$$\sigma_{\bar{x}}$$

### Probability of a MEAN rather than a SINGLE SCORE

**Example 1:** Children between the ages of 2 and 5 watch an average of 25 hours of TV per week with a standard deviation of 3 hours. If a sample of 20 children are selected, find the probability that the mean number of hours watched per week will be greater than 26.3?

Variables in this problem:

$\mu = \text{mean} = \underline{\text{_______}}$  \hspace{1cm} $n = \text{sample size} = \underline{\text{_______}}$

$\sigma = \text{standard deviation} = \underline{\text{_______}}$  \hspace{1cm} $\bar{x} = \text{mean of sample} = \underline{\text{_______}}$

### Central Limit Theorem

(Summary)

For ANY population of size “ $n \geq 30$ ” or for populations known to be normally distributed, the sampling distribution can be approximated by the normal distribution where:

$$\mu_{\bar{x}} = \mu \quad \text{and} \quad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

This means we can still use our table if we adjust our calculation for $z$!

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$
Example 2: Calculator Practice

Matching. Use the given information to compute a z-score.

- $\bar{x} = 29, \mu = 30, \sigma = 5, n = 35$
- $\bar{x} = 15, \mu = 12.5, \sigma = 3.8, n = 10$
- $\bar{x} = 110, \mu = 95, \sigma = 34, n = 20$

Example 3: A certain kind of bacteria exists in all water. Let $x$ be the bacteria count per milliliter of water. The health department has found that if the water is not contaminated, then $x$ has a distribution that is more or less mound-shaped and symmetrical. The mean of $x$ is $\mu = 3000$ and the standard deviation is $\sigma = 200$. The city health inspector takes 64 random samples of water from the city public water system each day. Assuming the water is not contaminated, what is the probability that $\bar{x}$ is less than 2925?

Example 4: The average sales price of a single-family house in the United States is $176,800. You randomly select 12 single-family houses. What is the probability that the mean sales price is more than $160,000? Assume that the sales prices are normally distributed with a standard deviation of $50,000.
Example 5: Is it unusual?

The weights of ice cream cartons produced by a manufacturer are normally distributed with a mean weight of 10 ounces and a standard deviation of 0.5 ounce.

a) What is the probability that a randomly selected carton has a weight greater than 10.21 ounces? Does this seem unusual?

b) You select 25 cartons. What is the probability that their mean weight is greater than 10.21 ounces? Does this seem unusual?

Example 6: Is it unusual? A machine is set to fill paint cans with a mean of 128 ounces and a standard deviation of 0.2 ounce. A random sample of 40 cans has a mean of 127.9 ounces. Does the machine need to be reset?