Computing Derivatives Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

General Rules

Definition of Derivative:
\[
\frac{d}{dx}(f(x)) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

Sum and Difference Rule:
\[
\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)
\]

Constant Multiple Rule:
\[
\frac{d}{dx}(k \cdot f(x)) = k \cdot f'(x)
\]

Product Rule:
\[
\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)
\]

Quotient Rule:
\[
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\]

Chain Rule:
\[
\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)
\]

Particular Rules

Power Rule:
\[
\frac{d}{dx}(x^n) = nx^{n-1}
\]

Exponential functions:
\[
\frac{d}{dx} e^x = e^x \frac{du}{dx}
\]

Trigonometric Functions:
\[
\frac{d}{dx} \sin(u) = \cos(u) \frac{du}{dx}
\]
\[
\frac{d}{dx} \cos(u) = -\sin(u) \frac{du}{dx}
\]
\[
\frac{d}{dx} \tan(u) = \sec^2(u) \frac{du}{dx}
\]
\[
\frac{d}{dx} \csc(u) = -\csc(u)\cot(u) \frac{du}{dx}
\]
\[
\frac{d}{dx} \sec(u) = \sec(u)\tan(u) \frac{du}{dx}
\]
\[
\frac{d}{dx} \cot(u) = -\csc^2(u) \frac{du}{dx}
\]

Logarithmic Functions:
\[
\frac{d}{dx} \ln(u) = \frac{1}{u} \frac{du}{dx}
\]

Inverse Trigonometric Functions:
\[
\frac{d}{dx} \sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}
\]
\[
\frac{d}{dx} \tan^{-1}(u) = \frac{1}{1+u^2} \frac{du}{dx}
\]

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Multiple Choice

1. A (1993 AB24)
\[
f'(x) = \frac{2}{3} (x^2 - 2x - 1)^{-\frac{1}{3}} (2x - 2)
\]
\[
f'(0) = \frac{2}{3} (0^2 - 0 - 1)^{-\frac{1}{3}} (2(0) - 2) = \frac{4}{3}
\]

2. E (1993 AB8)
\[
y' = \sec^2 x - (-\csc^2 x) = \sec^2 x + \csc^2 x
\]

3. E (1985 AB2)
\[
f'(x) = 4(2x + 1)^3(2) = 8(2x + 1)^3
\]
\[
f''(x) = 24(2x + 1)^2(2) = 48(2x + 1)^2
\]
\[
f'''(x) = 96(2x + 1)^3(2) = 192(2x + 1)
\]
\[
f''''(x) = 384
\]

4. B (AP style)
\[
h'(x) = f(x)g'(x) + g(x)f'(x)
\]
\[
h'(5) = f(5)g'(5) + g(5)f'(5)
\]
\[
h'(5) = (3)(5) + (-2)(4) = 7
\]

5. A (AP style)
\[
h'(x) = f'(g(x))(g'(x))
\]
\[
h'(4) = f'(g(4))(g'(4))
\]
\[
h'(4) = f'(3)(g'(4))
\]
\[
h'(4) = (-5)(9) = -45
\]

6. E (AP style)
\[
f'(2) = 2[g(2)]g'(2)
\]
\[
f'(2) = 2(8)(3) = 48
\]

7. E (1988 AB18)
\[
y' = 2 \left[ -\sin \left( \frac{x}{2} \right) \right] \left| \frac{1}{2} \right|
\]
\[
y' = -\sin \left( \frac{x}{2} \right)
\]
\[
y'' = -\cos \left( \frac{x}{2} \right) \left( \frac{1}{2} \right) = -\frac{1}{2} \cos \left( \frac{x}{2} \right)
\]
8. D (1988 BC10 appropriate for AB)

Using the definition of derivative, \( \frac{d}{dx}(\sin x) = \cos x \)

9. B (1988 BC3 appropriate for AB)

\[
f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} x^{-\frac{3}{2}}
\]

\[
f''(x) = -\frac{1}{2} x^{-2} = -\frac{1}{2x^2}
\]


Evaluate the equation for \( x = 1 \)

\[3(1)^2 + 2(1)y + y^2 = 2 \]

\[y^2 + 2y + 1 = 0 \Rightarrow y = -1 \]

Derivative

\[6x + 2x \frac{dy}{dx} + y(2) + 2y \frac{dy}{dx} = 0 \]

Evaluate \( \frac{dy}{dx} \) at \( (1, -1) \)

\( 4 \neq 0; \quad \frac{dy}{dx} \) does not exist at \( (1, -1) \)


\[\sin x = e^y \Rightarrow \ln(\sin x) = y \]

\[
\frac{d}{dx}(\ln(\sin x) = y) \text{ is } \frac{\cos x}{\sin x} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \cot x
\]

Method 1:

Use substitution to create $y$ in terms of $x$: $y = \tan \left( \ln x - \frac{1}{\ln x} \right)$.

Determine $\frac{dy}{dx}$ of the new equation.

\[
\frac{dy}{dx} = \sec^2 \left( \ln x - \frac{1}{\ln x} \right) \left( \frac{1}{x} - \frac{1}{x(x\ln x)^2} \right)
\]

Evaluate when $x = e$

\[
\frac{dy}{dx} = \sec^2 \left( \ln e - \frac{1}{\ln e} \right) = \sec^2(0) \left( \frac{1}{e} + \frac{1}{e(\ln e)^2} \right) = \frac{2}{e}
\]

Method 2:

Determine the derivative of each equation using implicit differentiation.

\[
\frac{dy}{dx} = \sec^2 u \frac{du}{dx} ; \quad \frac{du}{dx} = \left(1 + \frac{1}{v^2}\right) \frac{dv}{dx} ; \quad \frac{dv}{dx} = \frac{1}{x}
\]

Evaluate when $x = e$

\[
\frac{dv}{dx} = \frac{1}{e} \text{ and } v = \ln e = 1 \text{ which produces } \frac{du}{dx} \text{ to be } \frac{du}{dx} = \left(1 + \frac{1}{1^2}\right) \left(\frac{1}{e}\right) = \frac{2}{e}
\]

Finally $u = 1 - \frac{1}{1^2} = 0$ so $\frac{dy}{dx} = \sec^2(0) \left( \frac{2}{e} \right) = \frac{2}{e}$

13. A (1998 BC5 appropriate for AB)

\[
h'(x) = f'(g(x))g'(x)
\]

\[
h'' = f'(g(x))g''(x) + g'(x)f''(g(x))g'(x)
\]

\[
h''' = f'(g(x))g'''(x) + f''(g(x))g'(x)g''(x)
\]


\[
\frac{d}{dx} \sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}
\]

\[
\frac{1}{\sqrt{1-(2x)^2}}(2) = \frac{2}{\sqrt{1-4x^2}}
\]

15. E (1973 AB40)

\[
x \sec^2(xy) \left[ x \frac{dy}{dx} + y(1) \right] = 1
\]

\[
\frac{dy}{dx} = \frac{1 - y \sec^2(xy)}{x \sec^2(xy)} = \frac{1}{x \sec^2(xy)} - \frac{y \sec^2(xy)}{x \sec^2(xy)} = \frac{\cos^2(xy) - y}{x}
\]
\[ f'(x) = \frac{1 - 3e^{-3x}}{x + 4 + e^{-3x}} \]
\[ f'(0) = \frac{1 - 3e^{-3(0)}}{0 + 4 + e^{-3(0)}} = -\frac{2}{5} \]

17. D (2008 AB3)
\[ f'(x) = (x - 1)\left(3(x^2 + 2)^2(2x)\right) + (x^2 + 2)^3(1) \]
\[ f'(x) = (x^2 + 2)^2 \left(6x(x - 1) + (x^2 + 2)\right) \]
\[ f'(x) = (x^2 + 2)^3 \left(7x^2 - 6x + 2\right) \]

18. C (2003 AB92/BC92)
\[ f(1) = 1 \]
\[ f(4) = 4 + \ln 4 \]
\[ m = \frac{4 + \ln 4 - 1}{4 - 1} = \frac{3 + \ln 4}{3} \]
\[ f'(x) = 1 + \frac{1}{x} \]
Determine \( c \) where \( f'(c) = \frac{3 + \ln 4}{3} \); \( c = 2.164 \)

Free Response

19. 2003B AB6a
\[ f''(x) = \sqrt{f(x)} + x \cdot \frac{f'(x)}{2\sqrt{f(x)}} = \sqrt{f(x)} + \frac{x^2}{2} \]
\[ f''(3) = \sqrt{25} + \frac{9}{2} = \frac{19}{2} \]

2: \( f''(x) \) exponential \( y(10) \)  
3: \(-2 >\) product or chain rule error  
1: value at \( x = 3 \)
20. 2000 AB5/BC5

(a) \( y^2 + 2xy \frac{dy}{dx} - 3x^2y - x^2 \frac{dy}{dx} = 0 \)

\[
\frac{dy}{dx}(2xy - x^3) = 3x^2y - y^2
\]

\[
\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}
\]

(b) When \( x = 1 \),
\[
y^2 - y = 6
\]
\[
y^2 - y - 6 = 0
\]
\[
(y - 3)(y + 2) = 0
\]
\[
y = 3, \quad y = -2
\]

At (1, 3), \( \frac{dy}{dx} = \frac{9 - 9}{6 - 1} = 0 \)

Tangent line equation is \( y = 3 \)

At (1, -2), \( \frac{dy}{dx} = \frac{-6 - 4}{-4 - 1} = 2 \)

Tangent line equation is \( y + 2 = 2(x - 1) \)

(c) Tangent line is vertical when
\[
2xy - x^3 = 0
\]
\[
x(2y - x^2) = 0 \text{ gives } x = 0 \text{ or } y = \frac{1}{2}x^2
\]

There is no point on the curve with \( x \)-coordinate 0.

When \( y = \frac{1}{2}x^2 \), \( \frac{1}{4}x^5 - \frac{1}{2}x^2 = 6 \)
\[
-\frac{1}{4}x^5 = 6
\]
\[
x = \sqrt[4]{-24}
\]
21. 2007 AB3d

(d) \( g(1) = 2 \) so \( g^{-1}(2) = 1 \)

\[
(g^{-1})'(2) = \frac{1}{g'(g^{-1}(2))} = \frac{1}{g'(1)} = \frac{1}{5}
\]

An equation of the tangent line is

\[ y - 1 = \frac{1}{5}(x - 2) \]
22. 1980 AB6/BC4 (in 1980, each free response question was worth 15 points)

a) \[ 4x^3 - 10xy^2 - 10x^2y' + 16y^3y' = 0 \]
   \[ y' = \frac{4x^3 - 10xy^2}{10x^2y - 16y^3} = \frac{2x^3 - 5xy^2}{5x^2y - 8y^3} \]

b) \[ m = y'(2,1) = \frac{16 - 10}{20 - 8} = \frac{1}{2} \]
   \[ y - 1 = \frac{1}{2}(x - 2) \]

c) answers depend on answer from part (b)
   for \[ y = \frac{1}{2}x \] case, \((a, a), a < 0\)
   for any other line from part (b), \(a, \frac{a}{2}\), \(a > 0, a \neq 2\)
   \((a, a), a < 0\)

d) for \[ y = \frac{1}{2}x \] case, \(\left( a, \frac{a}{2} \right), a < 0\)
   for any other line from part (b), \((-2, l(-2))\)

Differentiation

-3: no product rule
-2: other calculation error
Solves for \(y'\)

\[ y' = \frac{1}{2} \]

Linear equation

\[ 0/5 \] if not linear

Correct points

(0, 0) for any other line from part (b) receives \(1/2\)
\(0/2\) all other points

Correct points

\(0/2\) all other points
23. 1978 AB2

a) \( h(x) = f(g(x)) \)

b) \( h'(x) = 2(1 - \ln x) \left( \frac{1}{x} \right) \)
\[ = \frac{2}{x} (\ln x - 1) \]

c) \( h''(x) = \frac{2x \left( \frac{1}{x} \right) - 2(\ln x - 1)}{x^2} \)
\[ = \frac{2}{x^2} (2 - \ln x) \]

d) No point values provided.